Andrea Paolini

IMPRINTING THE REALITY

SPECULATIONS ON PHOTOGRAPHY

For graduate students, photographers and all people interested in scientific speculations and art

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PREFACE (limits of photography and of this book)

0.0 Photography and Arts, “real reality” and “virtual reality”

Photography without post-processing[1] is like cinema (and theatre)[2], that is, it memorises the “existing matter”, eventually adding fiction, that is, composition of the scene[3]. Instead, writing and drawing/painting (ideally except “historical” writings and “drawing/painting from life”), and also sculpture, can “*invent*” from “nothingness”[4], can be “intellectual constructions”[5] that arrive at “material constructions”[6]. Animation films[7] and virtual reality are the same as painting and free/creative-post-processing-photography[8], but extended in time domain, with the virtual reality that can be multi-sensorial (sight, earring, smell...); so they are the most “free” and “complete” arts.

This book covers only photography without post-processing, so only means to register/reproduce visual[9] components of “reality”/real-scene[10] (“experience”), i.e. the reproduction of something that exists [and that is material], what we could name: “real-reality”. But the aim to reproduce reality in a perfect way, apparently typical of only this limited perimeter, can be ascribed to the field of the technical “domination”, and is an

---

1 We are meaning a retouch to intentionally change the reality imitation; it can be made by any technicise (manually, the “old” photo-montage, in analog photography, by the aid of a software in digital photography).

2 but limited to their visual component: their words, poetry and music are not of concern

3 if the scene can be changed (as in nature painting); in cinema and theatre there is also recitation. A “perfect” reproduction of reality (by photography [or “by painting”]) is not art (see also Note 11), so, paradoxically, photography and life painting, forcing the spectator to see the “scene” from a particular $P.o.V.|_{3D\ scene}$ go “towards art”.

4 But on the possibility of creation, see also: A. Paolini, *On the immortality in the Arts [which is also the problem of the conservation/existence/definition of the memory] (with a note on memory, time and life),* Mathforlife.net Project (https://mathforlife.net/).

5 Comics with their stories, in particular, as also animation films, cut himself off from reality.

6 Physics says that anything is always material, but in the case of visual Art, and virtual reality (see in the following), the piece of Art creates, by a sensorial illusion, the illusion of another, “more material”, reality.

7 In animation films the scene is seen from the *P.o.V.* chosen by the director; in virtual reality the *P.o.V.* is not decided, so it resembles to an “invented” “cinematic” (time changing) holography.

8 the “freedom” of changing the representation of the “reality”.

9 In fact, to reproduce reality all five senses would be necessary: sight, frozen in photography and varying in time in TV and cinema, smell (present in some experimental cinema; it is not difficult from a technical point of view, with modern artificial smell and modern means for smell reproduction) and touch. The touch is difficult to be reproduced, because the “camera-man” should use a tactile overalls with sensors for touch pressure measurement and the spectator wear a special overalls with pressure actuators to reproduce it: that's made in virtual reality.
Completion of sight with hearing, as well as with other senses, is “easy” because hearing is uncorrelated from sight.

10 with the use of a post-processing software, that conceptually introduces invention (like painting and virtual reality) only to correct fidelity errors.

unavoidable statement/base for any *conscious* act, in particular for the "creation" of any piece of art[11].

0.1 The further limitation of this analysis

Photography is the theoretical extraction of the purely spatial components from the (dynamical) spatio-temporally reality[12]. So photography[13], against video, is not interested to the "reproduction" of the entire reality[14], doesn't matter the speed of the world (this is a "philosophical"[15] choice), but in reality space coordinates are influenced by speed. So photography would take into account of these influences.

There are two effects of speed on spatial coordinates:

- The relativistic spatial deformation[16]: an anisotropy spatial distortion (for very far (3D) objects it could be similar to a rotation[17]).

- The motion blur that can not be eliminated according quantum physics[18] [19].

All photographers know very well the motion blur that is relevant with fast objects: they try to use fast objectives and strong illumination/very sensitive photoreceptors, adding the use of flash for not too far scenes.
To avoid this effect we should make the *measure* in an infinitesimal time, so:
1. The shutter time should be infinitesimal[20]. If the duration is infinitesimal, the constancy of c implies infinitesimal spatial dimensions for the shutter.

---

[11] The prerogative of the [conscious] artist is the intentionality, that needs the illusionistic "domination" of the technique and (an even more greater illusion) of the representation/world; otherwise the piece of art is "only" a selection from the chance (this, only apparently, needs more luck), i.e. the author is "only" a critic of arts, but that is not inferior to the artist. The supposed "domination" can give the right to switch from 3D to 2D photography, and from colour to black&white photography, not as an obligation due to a technical inability, but as a poetical choice.

[12] Objects always move respect to the photographer.

[13] We take into account one single photography (sequence of photographies are used for movement estimation and analysis (as eye+brain make), but this method is outdated by video frame analysis).

[14] Video can show, besides the spatial deformation, also time dilation, so the slowdown of speeds, i.e. of the life that flows slowly there.

[15] Photography *would* like to freeze time (because we don't eliminate time, and not eliminate time, limited to restricted spatial regions, but that we can chose). But this freezing contains inside the trace of the speed (of the time), see its two spatial effects in the following. With statical scene hypothesis the intention is the same (but simplified), also if it could reveal the intention to eliminate time (see Note 24).

[16] See: A. Paolini, *Relativistic notes*, https://mathforlife.net.

[17] If the object is not seen exactly along the relative speed direction, but this is always the case because its finite angular extent implies that a non-orthogonal component of the relative speed vs. the photosensitive element is always present. This is true for the front face (2D spatial contraction) and for the volume (3D spatial distortion).

[18] See: A. Paolini, *cit*.

[19] The relativistic spatial distortion at the previous point is due to only speeds. The spatial distortion due to the spacetime curvature is visible only by a time variation of $T_{\mu\nu}$. This should not be visible according to relativity in the pure spatial domain of one single-photography, but, due to this quantum effect, in the motion blur the spacetime curvature is visible (in general a negligible effect also for very distant (fast) objects).

[20] This is Einstein's hypothesis for the measure of distances.

2. The photoreceptor should be sensitive to an infinitesimal energy[21]

In the Einsteinian fallacious hypothesis of the continuum (spacetime continuum, but also of Tmu-nu, the two are linked), position (sharp photography) and speed (more than one photo, in sequence) are determinable with precision at will . So motion blur can be, theoretically, eliminated.

In quantum physics space and time can not be "separated" (better: space and speed in our case), because the action (energy x time) can not arbitrarily diminished, so point 2 can not be implemented, and to use fast shutter time (to have sharp photographies, so good precision on the position) we are obliged to use big powers (a lot of photons), perturbing the object speed, with a low precision on the speed[22] (in the sequence of photos), particularly if the object to be photographed is in microphysics[23].

In this book, we neglect speeds, so we make the supposition that the scene is a *statical* scene[24] or that the speed doesn't influence spatial coordinates, that is a pre-relativistic and a pre-quantum approach[25] (or we have to precise that these are negligible "effects"[26]).

So with no speed the treatment is simpler[27].

0.2 Photography, eye, vision

Any optical support that visually reproduces the reality must pass through the human eye[28] that implies a degradation (corruption) of the (optical) power of the real scene that arrives to the brain (so that is the "image" of the reality). If the reproduction by-passes the optics of the eye (by a so called artificial eye), a person can see, if the artificial eye has an optical performance better than the natural eye of that person (that can also have, for example, optical defects like myopia) even better; this is true if connections to the optical nerve or to the visual cortex are as good as the human ones. In any case, the visual cortex performance (we can not increase the neuron number) would be the final limitation.

---

[21] Or the energy used for the measure should be infinite, but this is discarded also by Einstein, quantum physics is not necessary.

[22] If photons for measuring are orthogonal to the relative speed of the object, the indetermination in the speed is along this orthogonal direction, so the motion blur is along this orthogonal direction.

[23] The same limitation is present at the level of the shutter itself, that, just, as said, must be miniaturised (for example at solid state, where, especially, must subject to quantum uncertainty).

[24] This is always an idealised supposition, also if we are in front of a very quiet landscape, because "only space" would exist. But "only space" is an erroneous concept, because without time also space would disappear (only "to be" would exist). Because we don't photograph the "*entire* space" (space would disappears), but only landscapes (at different times), really time doesn't disappear (the scene is not really static (not for all eternity)).
Note that "static" means also that the object doesn't relatively move also during the measure (photography uses photons) (so the motion speed quantum effect is not present).
In the case of statical scene*s*, photography is useful just more for another of its scopes: to transport a place in another place.

[25] The same simplification is used by perspective theory (that in any case treats "statical" scenes) and also by imaging/geometrical optics. Comics and some "futuristic" painting try to represent the speed effect of the motion blur (but not the relativistic distortion considering that it is not perceptible).

[26] The relativistic spatial distortion was not noted before and it is very hard to be appreciated (a photo of the faster (terrestrial) object that we can easy photographed steady and in movement, an airplane at 1000 km/h: its spatial contraction ($\gamma = (1- 8 \text{ x } 10^{-13})$) is much smaller than the granularity of the photoreceptor and the distortion of objectives, also using telephoto objectives). The motion blur was thought to be theoretically eliminable before quantum physics, but it can no, also if technical progress of shutters and photoreceptors can make it negligible for "low speed" objects.

[27] The relativistic spatial distortion becomes null (but generally it is not visible). The greater visual approximation is about the motion blur that practically, without the search for ultrafast microscopic shutters and ultrasensitive (to single photon) photoreceptors, can be visible.
An important consequence of this assumption: also the statistical uncertainty disappears because without time the ergodicity assumption that is always feasible disappears.

[28] Here, only the optics of the human eye (w/o retina, optical nerve and visual cortex contributions).

0.3 The "dream" of the "perfect" photography

The purpose of photography is, mainly[29], to evoke, seeing a photography from a fixed distance and given environmental illumination, the same sensations that the photographer felt when he took that photography; to obtain this, the spectator must see *exactly* what the photographer saw[30]. You can live again what you lived when yourself took the photography, or you can live what other people (the photographers) would like to make you live (as, for example, to see wonderful landscapes that you could never see without to make a (real) trip).

This is not possible[31].

Nevertheless, we can reproduce, for a normal sight spectator, "exactly" what a person with imperfections saw.

---

[29] We exclude, as said, post-processed photos, so "artistic" photography.

[30] About the photography not relied to the reproduction of the eye sensation: photography from impossible *P.o.V.*s (mainly macro and micro-photography, but also from unreachable *P.o.V.*s), freeze of too fast movements (so photography becomes also chrono-photography for scientific analysis, and permits the rallenty), photos with too short *d.o.f.* (some objects are out of focus) (but in this case refractive defective eye can give the same effects) and images not visible by nuke eye (in a quasi-complete dark or using infrared or astronomical photography or, again, macro and microphotography). In particular, the peculiarity (in comparison to the painter) of the photographer that photographs dynamical scenes (that takes "snapshots" as still-life photos) is his ability to catch the instant: he is an inquirer of time (space-time) and not only of space. In this book this peculiar aspect of photography is not treated in detail (see Par. 0.1).

[31] It is impossible to make this "perfectly": this book establishes the "truthfulness" of this statement, "except" when the person has a defective sight and uses ophthalmic lenses only after, for seeing the photography of the scene.

The "perfect copy" is an oxymoron, see: A. Paolini, *On the immortality in the Arts [which is also the problem of the conservation/existence/definition of the memory] (with a note on memory, time and life)*, (https://mathforlife.net/2018/07/01/dellimmortalita-nellarte/), cited; besides, a representation of reality is not reality, that must be personally *lived*, *hic et nunc*.

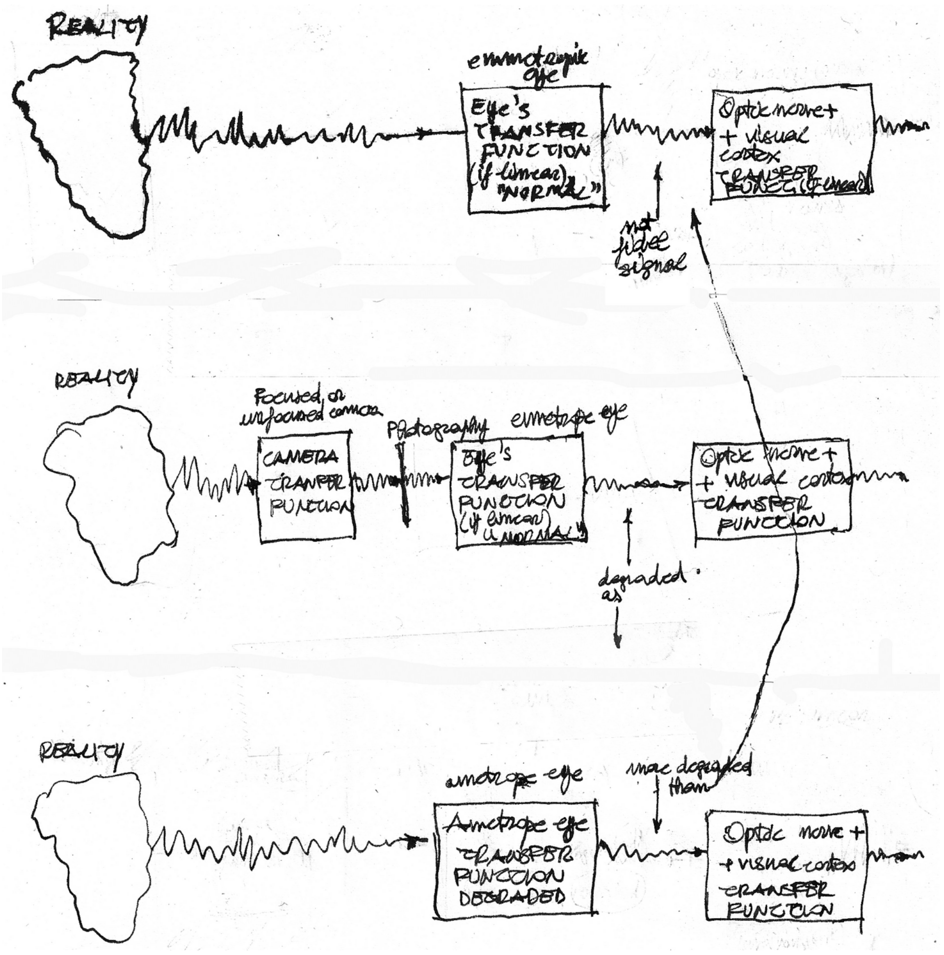
REALITY
emmetropic eye
Eye's TRANSFER FUNCTION (if linear) "NORMAL"
Optic nerve + + visual cortex TRANSFER FUNCT. (if linear)
not fidele signal
REALITY
Focused or unfocused camera
CAMERA TRANSFER FUNCTION
Photography
emmetrope eye
Eye's TRANSFER FUNCTION (if linear) "NORMAL"
Optic nerve + + visual cortex TRANSFER FUNCTION
degraded as
REALITY
ametrope eye
Ametrope eye TRANSFER FUNCTION DEGRADED
more degraded than
Optic nerve + + visual cortex TRANSFER FUNCTION

Fig. 0.1 - The dream of the perfect photography: the ametropic eye vision reproduction.

INTRODUCTION: Optical information and its memorization

1. Information to be memorized and methods

The optical information of the changing (in space and time) scenario (of the points of the objects that interact with light, and of the points of the air) is constituted by its spectral composition and its luminous intensity[32] [33]. If this information is memorized for all the space around the scene, it is possible to turn around[34] the memorization support/image and, seeing it with one or two eyes from different points of view, to receive the "same" optical sensation as watching to real scene from the same points of view (as a note, the observer (the last receiver is always a human receiver) is not able to "define" a 3D scene except that by projection(s), using as projection point(s) (*P.o.V.*$|_{3D\ scene}$) his one or two eyes[35]).

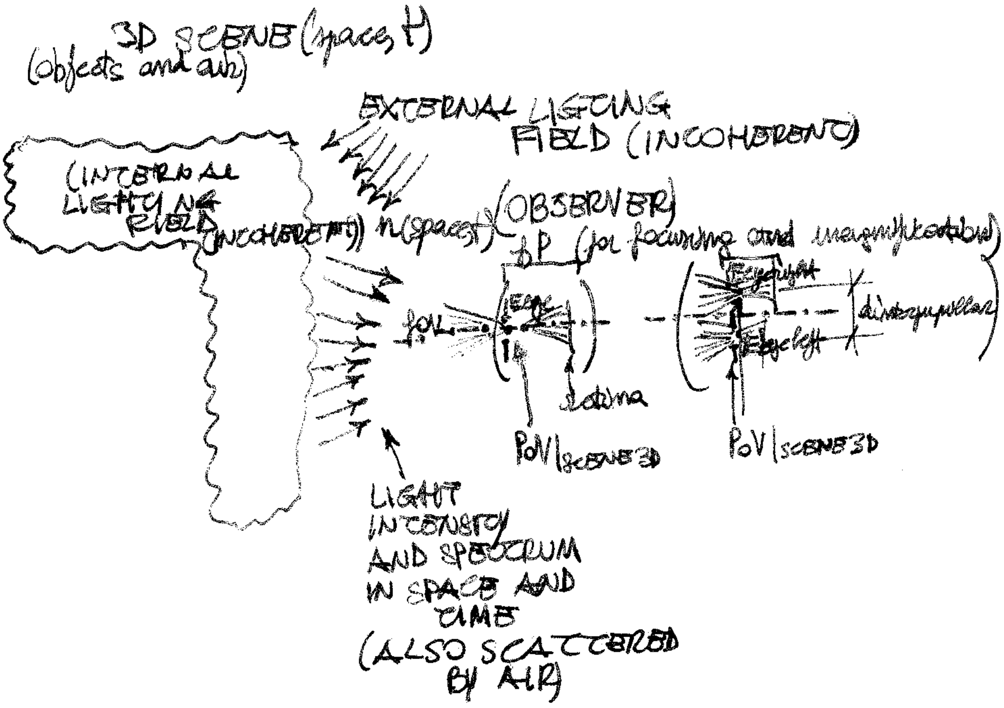
3D SCENE (space, t)
(objects and air)
EXTERNAL LIGHTING FIELD (INCOHERENT)
(INTERNAL LIGHTING FIELD (INCOHERENT))
n(space, t)
(OBSERVER)
ΔP (for focusing and magnification)
f.o.L.
Eye
retina
Eye right
Eye left
distance pupillar
PoV|scene 3D
PoV|scene 3D
LIGHT INTENSITY AND SPECTRUM IN SPACE AND TIME (ALSO SCATTERED BY AIR)

Fig. INT.1 - Optical information and its memorization.

---

[32] We are considering an incoherent illuminating field.

[33] Or, equivalently, by the intensities of three frequency optical wavefronts (RGB components) (final detection by the eye is considered).

[34] Supposing that the entire wavefronts have been detected.

[35] So, generally, it is convenient to express the positions of the "emitting" points (the "origins" of the spherical wavefronts) of the 3D scene in polar coordinates, with as a pole the eye or the two eyes of the observer.

The information has 3 degrees of freedom x 3 (due to RGB composition), so 9 degrees of freedom in total. Due to the independence of each colour from the others[36], we will treat only the intensity information, for which we need 3 degrees of freedom to memorize it[37].

The three degrees of freedom of the intensity information are captured by the following two methods:

1. Making reference to Fig.INT.2, given anyone illuminating field, the imaging optical system can be used together a **3D** transparent photosensitive/memorization element able to react to and to store intensity, because imaging optical systems are able, ideally[38], by actuating a projective transformation, to conserve the 3D information of the scene, also if loosing the scale information[39] (that, nevertheless, can be recovered (to restore all 3D information (the wavefront)) by knowing the cardinal points of the optical system and one distance (of a point in image space or in object space)); nevertheless, to see the image by nuke eye another operation is necessary: to apply to the distorted 3D image[40] a compensating distortion, by calculation[41] (the use, for example, of a simple magnifier (a single converging lens) to recover the distorted 3D image could give a good 3D image restoration, but only when the 3D image is seen along its optical axis of deformation).

---

[36] Only in the simplified hypothesis that all systems are no-dispersive systems, see APP.5.

[37] Here we are speaking about degrees of freedom and not information quantity (entropy of information) that depends on the discretisation (resolution) of our memorizing means. Resolution should be, theoretically, infinite (and not simply higher than the eye resolution capability) to give to the eye a perfect reconstruction of the reality, because if it is true that eye has a finite resolution, so can treat a finite quantity of information, although the reproduction system has always a non-unitary transfer function, in other words the resolution of the reproduction has always a negative impact.

[38] In the ideal case of perfect imaging, i.e. in paraxial optics (see APP. 2).

[39] So we can use homogeneous coordinates

[40] Nevertheless, also with the favourable case of an optics of revolution (a simple lens or an optical system composed by centred lenses or centred revolution systems) the 3D shape of the object is not maintained (generally it is mignified) because of the different transversal and longitudinal magnifications (see APP. 2); in particular, with an optics of revolution around z, z has a different scale factor ($-n'/nk^2$) respect to x and y (k). This is the reason why in this method the (optical) axis of the axial symmetric projection (and not only the projection points: the front and rear nodal points) must be specified [in general the focal planes and the projection tensors] (see this and the following pictures) because the projection is not an “isotropic” projection (magnifications depend where the axis is (M_{transv} is different from M_{long} , except at he distance for which $M_{transv} = M_{long} = 1$)) so the detection/decoding must be made along the right direction.

[41] This compensation (the longitudinal factor must be the square root of the transversal factor) can not be carried out by the nuke eye.

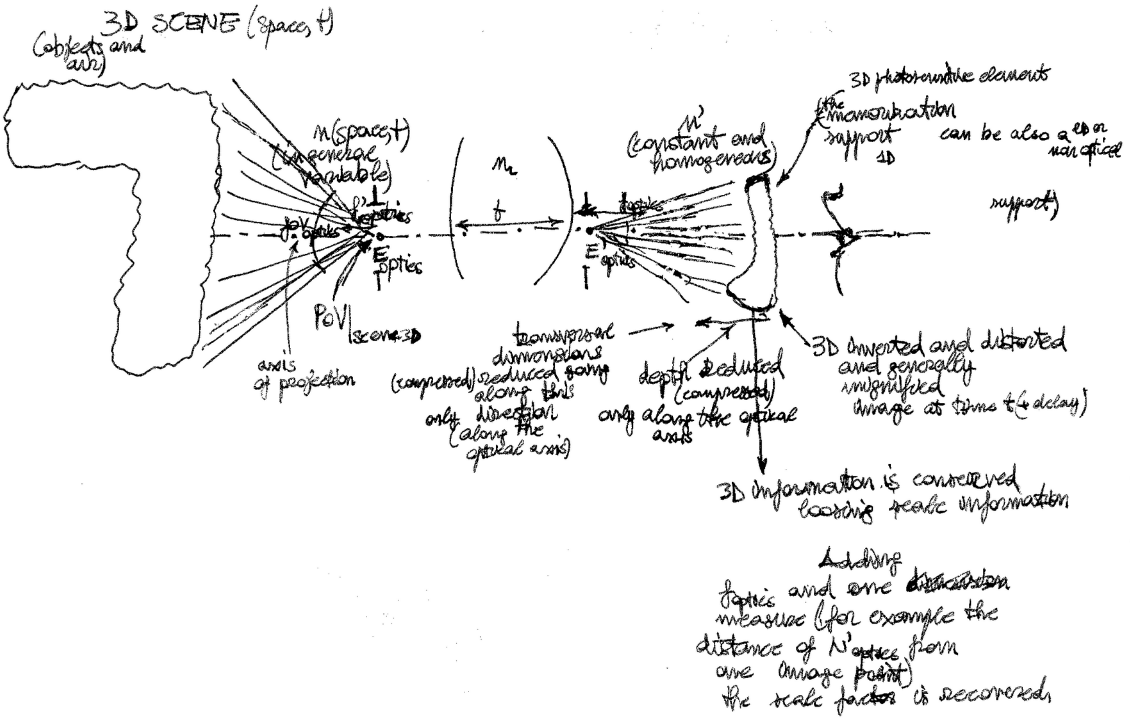
3D SCENE (space, t)
(objects and air)
n(space,t) (in general variable)
f'⊥ optics
f⊥ optics
F optics
PoV|scene 3D
axis of projection
n₂
t
n' (constant and homogeneous)
f optics
F optics
3D photosensitive elements
(the memorization support 1D can be also a 2D or non optical support)
transversal dimensions reduced going along this direction only (along the optical axis)
(compressed)
depth reduced (compressed) only along the optical axis
3D inverted and distorted and generally unmagnified image at time t (+ delay)
3D information is conserved loosing scale information
Adding f optics and one distance measure (for example the distance of N' optics from one image point) the scale factor is recovered,

Fig. INT.2 - An optical system actuates a 3D projective transformation but with different transversal and longitudinal factors (magnifications).

2. A 2D photosensitive memorization support[42], plus a coherent illumination field[43] (3 total degrees of freedom: 2D intensity + phase of the interference field (so 2D intensity, as above, + 1 (= scale factor or one distance) $\Longrightarrow$ the 3D shape with also the scale of the object are conserved) (see Fig.*INT.3*).

---

[42] As the one used in “standard” cameras, able to store only intensity.

[43] constituted by an illuminating coherent wave plus an additional coherent reference wave to code the phase of their interference field in intensity

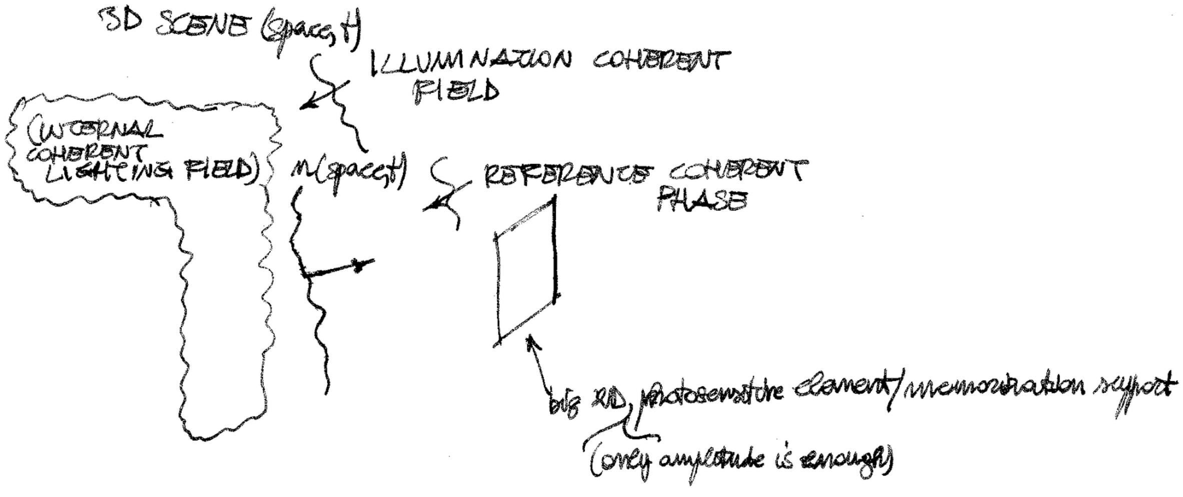
3D SCENE (space,t)
ILLUMINATION COHERENT FIELD
(INTERNAL COHERENT LIGHTING FIELD)
n(space,t)
REFERENCE COHERENT PHASE
big 2D, photosensitive element/memorization support
(only amplitude is enough)

Fig. INT.3 - Holography

Taking also into account that the goal is to recover the entire scene information but also to present it directly to the eye, so without distortion and with real (1:1) scale, the problems of method 1, after applied the deformation, are:

1. The (transparent) optically sensitive mean must be 3D
2. The 3D optical reproducing support (made by the lightest material as it is possible!) must be scaled to real dimensions (1:1), after have known cardinal points and one distance

The problem of method 2 (holography) originates from the fact that the illuminating field(s) is not the given, external, natural one (in general the sun light, with its variable spectrum, direction and intensity), but it must be a coherent, artificial one[44] that forcedly is spatially limited[45] so we can not light the entire landscape by a laser, nevertheless for far objects, 3D/wavefront information (scenario depth information) is not very important, due to eye binocular errors, and grace to the fact that we can not turn around the objects. For these reasons, holography is limited to studio photography[46] where it is the best technique to conserve information, more “impressive”[47] than “bi-objective” photography[48] because the spectator can move around the image.

---

[44] the coherent reflectivity/air-transmission of the scene must be supposed isotropic do not need infinite artificial illuminating fields

[45] Besides, because there is no optics with its mignification, the photosensitive 2D element must be as big as the optics of the other methods, to collect the same amount of field.
Without projection by optics, the radial dimension of the “observer centred” polar coordinates is now conserved, so the field can be better expressed in “orthogonal coordinates”.

[46] Where in general an artificial illumination is used,

[47] The fact that “impression” is correlated to the realism of the image reinforces the importance of an exact reproduction of the reality.

[48] See next method 3. “Bi-objective” photography neglects eye’s convergence angle and focusing, but also considering them, it is point-of-view-dependent, an “artistic” prerogative (a “limitation” to the freedom of the spectator that must “obey” to the “will” of the photographer).

With method 1 it is easy to add RGB information using “colour” photosensitive elements, to arrive to the 9 degrees of freedom, whereas with method 2, other than clout photosensitive elements, three lasers of different colours are necessary.

3. The problems of methods 1 and 2 are avoided by the bi-objective[49] method that uses the given incoherent illumination and two 2D projections[50] [51]. Total degrees of freedom: the two couples of 2D intensities can give the 3 independent full data[52], as desired, if the positions of the 2 projection points[53] [54] (their transversal distance (separation)[55]) and the projection parameters are known, by an image processing; and, directly by the eyes, if the two groups of projection parameters are the same of the eyes[56], also if decoding by the eyes is not exact because the inter-objective distance seldom is exactly the same as the interpupillar distance of observer of the photography i.e. of who tries to observe like the photographer saw. Bi-objective photography has the same principle of the binocular eye vision, but, differently respect to real vision, the “redundant” information given by the vergence angle and by the focusing degree is supposed do not influence the reproduction[57], so not used in normal applications (generally the two 2D images are not presented to the eyes at the right longitudinal distances to allow for the correct eye’s convergence and focusing).

---

[49] during the shot (coding phase), in such a way that, during the vision, this method can be directly used by the nuke eye. Binocular method uses a similar mechanism, but with redundant information for checking, that is not restored by bi-objective method during the vision of the two photographies, see later in this paragraph.

[50] The 2D supports (surfaces) generally are the same because the projections overlap (giving possible problems of linearity/dynamical range of the support).

[51] For example, as we saw, by the two “3D methods” above, a 3D reconstruction is impossible for big scenes as landscapes, because of too much information and energy demanded for it. On the contrary, bi-objective photography is applicable to big scenes because it is only a partial projection of the scene according to two *P.o.V.*s.

[52] Despite the photography using method 3 could recover, by *calculation*, the 3D information constituted by the original wavefront-phase/point-absolute-positions, generally it is simply used to present to the eyes the two images (the brain that is not able to reconstruct the wavefront (perhaps except when someone tries to image the scene as seen from different points of view)), so the decoded information is the one of as a scene that is seen from only one point of view (constituted by two eyes), the one(s) of the photographer.

[53] Neither the axes of projections, nor the two (or one) surfaces (generally planes) on which the projections are executed.

[54] To recover the 3 degrees of freedom as in methods 1 and 2, we collect more information than necessary: two times the 2D information of angular positions of all the points, with a one dimension correlation between them (the inter-objective/interpupillar distance). Note that also mono-objective photography and monocular vision (analysed in the following paragraph) permit to arrive to *calculate* the longitudinally/radially information (so the 3 degrees of freedom), by the 2D information of angular position of all the points and, as it is potentially possible, by the defocusing amount (see the following Par. 2).

[55] If projection points are at infinite (the so called telescopic projection in optics), as also in orthogonal projections (that maintains the scale), the 3D complete information is derived without to know the distance between the *two* projection points/axis.

[56] Eyes are considered to have the same projection parameters, so the camera has identical objectives (the two scale factors, k_1 and k_2, are the same).

[57] The brain uses also the vergence angle and (probably) also the focusing information of the eye crystalline lenses.

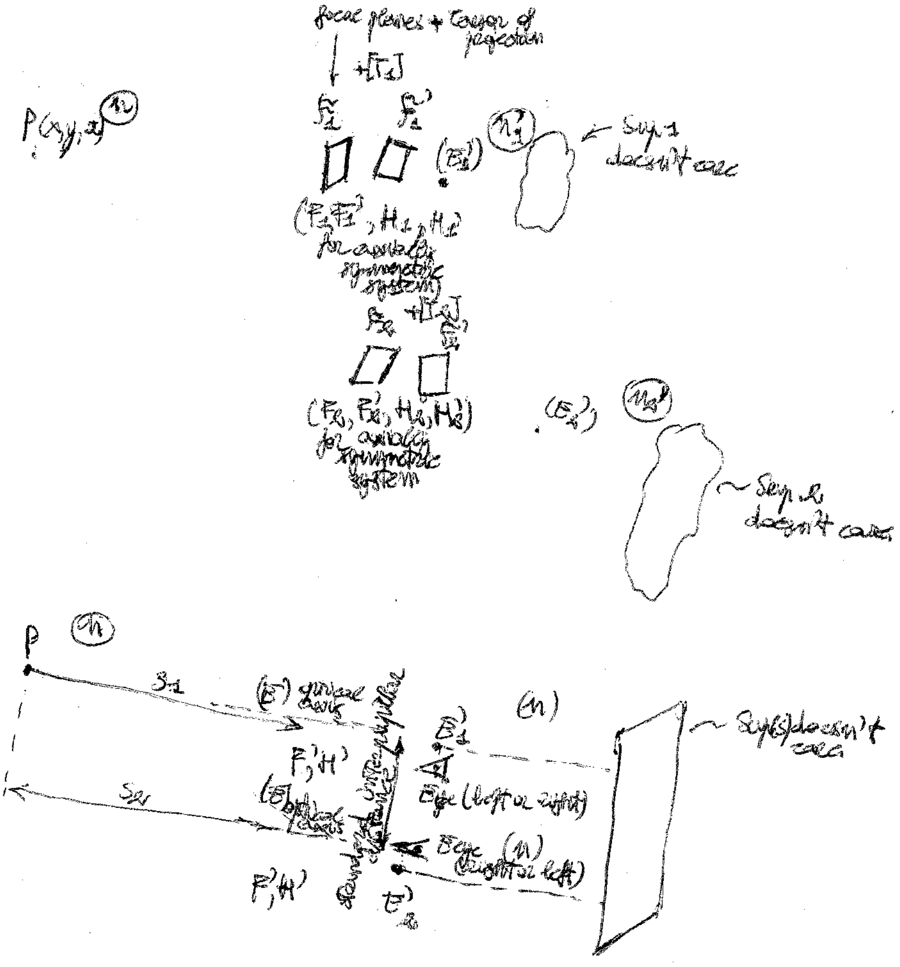

Fig. INT.4 - General bi-objective method and its variant (axially symmetric with identical projection parameters) for a direct decoding by the eyes (but with additional distance/s evaluation methods (vergence and focusings) in general not respected; interpupillar distance is a priori estimated (standard value))

2. Reduction of the memorized information (blind in one eye)

The general theory of incoherent photography using two 2D separated projections (bi-objective method, see point 3 of previous paragraph) becomes a restricted theory for the particular case of mono-objective photography and of monocular vision at which, for far objects, they tend towards: the mono-optical method, so mono-objective photography and drawing/ painting that use the perspective theory, can give a good approximation of the binocular vision, especially where this one looses accuracy, so, as said, in case of a *far* scene (all the objects are far away), so they can be seen by two eyes with good results[58] that the weaknesses (that disappear for far objects) of monocular vision can permit[59].

With this precondition (the objects are far away), the advantage that gave impulse to 2D imaging (mono-objective photography and (2D) painting/drawing) respect 3D reproduction (sculpture) is the *energy*[60] saving that the weaknesses of monocular vision permit: the scale can be reduced and the shape in the third dimension can be avoided, with a greater advantage in case of far objects.

We analyse more in detail these weaknesses of monocular vision which we can take advantage of:

- To scale: one eye is not able to well estimate the distances, so it can be profit by, to cheat the single eye, by constructing 3D models to scale or a (single) 2D painting/drawing/photography to scale[61]. In the case of the 2D reproduction, the nearer object of the scene is far from the projecting surface (in accordance to the rule to have all the objects far from the $\text{P.o.V.}|_{3D\ scene}$ to have a good impact/approximation also seen by both the eye the picture).
  On the contrary, if we try to diminish the energy advantage of the scale, to improve the truthfulness of the nearer object, and we represent it to scale 1:1[62], in any case this object in foreground must be far from the observer of the picture (or this object must be very slim), so the advantage of the real (2D only) dimensions of one object is diminished by the limits of the monocular vision (the perspective illusion of its third dimension would be corrupted proportionally): the result would be a big picture[63] to be seen from a far distance (with a high expense of energy/fatigue).

---

[58] Otherwise we can obtain nearly perfect results seeing the 2D photo or the 2D drawing/painting by only one eye but respect to a vision of the scene with only one eye (we are perfect but one-eyed), but we have to accept the fact that we see the scene and the reproduction with only one eye.

[59] Psychology can do acceptable (giving a similar illusion) to see the 2D image also by both the eyes.

[60] But, for perspective construction of invented scenes, with an increase of the information entropy.

[61] If they are seen, as usual, by both the two eyes, they are recognised as simply models to scale and paintings/drawings/photographies to scale (in this case also simply 2D models), but in particular only at short distances. In the case of the 3D little model, the psychology doesn't help as with 2D images because, paradoxically, in this case the artificial reproduction nature is a clear complicity from the first sight (the only way to obtain a very realistic result is to take a 2D photo or to shot a 2D video of the miniaturized 3D model, so the binocular vision must see a monocular reproduction, see cinema effects; see also the 2D shifted reproduction of dioramas a and of theatrical scenography that are 3D or semi-3D but seen from very far distances at theatre).
Note that 3D photography (method 3 of the previous paragraph) is nearly insensitive (not corrupted) by scale [I discover the scale of the each of the two reproductions seen each one with one eye].

[62] Tin photography, never the nearest "object" can "touch the photography" because a zero focusing distance is not achievable. A solution is not possible: magnifying "more than reality" the photography doesn't solve the problem. See Note 71.

[63] Except when the foreground object is little.

- Third dimension lost: 2D imaging (mono-objective photography and (2D) painting/drawing) can neglect, respect to 3D reproduction (sculpture), to shape the given projection surface, so the compactness of the model to scale further increases and it can be attached to the wall (using a flat projection surface). Nevertheless, paradoxically, for an invented scene, we must know the entire 3D information to apply the perspective calculations, so we pay by "information" entropy[64] [65] instead of "physical" energy to shape the third dimension[66]

  This advantage can be added to the previous one, so the reproduction can be to scale and flat.

Mono-optical method (mono-objective photography / monocular vision)[67] extracts a 2D information according to a single projection (respect to the projection point: $P.o.V.|_{3D\ scene}$) on a photosensitive surface (generally a plane) (the photography/drawing or the retina); the single image of a photography/drawing, if seen by only one eye in the good $P.o.V.|_{photo}$, it is similar (so except for a scale factor[68]) to what would be seen by the observer's unique eye in the $P.o.V.|_{3D\ scene}$. If additional information constituted by de/focusing degree, in conjunction with cardinal points and aperture stop size and position[69], is taken into account, scale and depths of the scenario can be *mathematically calculated* (i.e., adopting polar space coordinates, radial dimension can be recovered[70]). Except this theoretically case, commonly it is the "(linear) perspective rule"[71] that can help to estimate (by seeing[72] photos or by seeing directly the scene) depths/distances of the "real"[73] scene, basing on experience by supposing to know the (transversal) dimensions[74] of objects, being more reliable[75] when at least two ("identical") objects at different distances are present in the scene because one absolute distance estimation plus a relative comparison of subtended angles can be made more easy[76] than the absolute evaluation of each distance by single angles.

---

[64] We must know how the eye must the projection, i.e. simplify the 3D information. In 3D photography (third method in the previous paragraph), the more complicated calculations permit do not loose scale also if the two 2D images are to scale.

[65] Except when the calculations are made automatically by the camera, that in any case requested for "physical" energy for its construction and "entropy" energy for its design.

[66] Using perspective rule, the scale factor is lost, so third dimension, as the other two ones, are relative ones.

[67] If not otherwise specified, in the following paragraphs mono-objective photography case will be treated.

[68] So the scale factor k of only a 2D homogeneous coordinate system $[x\ y\ k]$ (here for a projection along z axis) is not defined .

[69] See *d.o.f.* definition on Par.A.1.3.1.1.1: deriving $s_{focused}$ (so the scale) by $s'_{focused}$ and f', and knowing E and P_M, from the defocus amount ($\Delta\beta_{blur}$) it is possible to derive all the depths of the object. Scale can be also determined by knowing one length in the object space and its angular extension.

[70] With scale factor k and z defined, the 3D homogeneous coordinate system $[x\ y\ z\ k]$ becomes a 3D coordinate system $[x\ y\ z]$.

[71] Other than linear perspective, helps for estimating depths are: aerial perspective, shadows, etc. . In any case, visual perception is not only optics, but must be treated, finally, by psycho(physio)logy.

[72] If by two eyes, all the following considerations are approximated on the basis of the object distance.

[73] But by perspective it is not possible (see Note 45) to resume scale, so to say to discover if the depicted scene was "real" or was simply a (mignified) model. The same for the reproducing scale of the photo/painting: also this one can be changed without perspective effects.

[74] Theoretically, the absolute distances and depths estimations could be done by absolute subtended angles but really *knowing* (an information that is not in the perspective) the objects absolute dimensions, so *knowing* the possible scale factor of each object.

[75] Except, again, for the distinction between the "real" scene and the model: in this case the ratios, as the single angles, are identical.

2.1 Projective transformations: coding and decoding of image(s). Perspective.

2.1.1 Coding and decoding of image(s)

Holography and 3D-support memorization have "no coding"[77] of the image which, consequently, can be directly seen by the eye as it is in reality[78].

On the contrary, in bi-objective and mono-objective photography there is a "coding" of the image, but that can be decoded directly by the eye/brain if the image is seen according to certain geometrical conditions (respecting the Point(s) of View for the image(s)).

2.1.2 Depth determination in monocular vision and mono-objective photography

2.1.2.1 The linear perspective effect ("Natural" and "artificial" perspective)

In the (nuke) eye, eye optics actuates a projective transformation[79] that projects images onto the retina (that is a (curved) surface, so a 2D geometrical form, i.e. one information ("longitudinal/radial") is lost[80]). So it seems that "miraculously" eye optics actuates the calculations of the perspective rules discovered by the renaissance painters (the "artificial" perspective), but it is the contrary: the "artificial" perspective rule discovery was an unconscious and unintentional discovery of the optics of the eye (of the "natural" perspective) (and that it miraculously carried out automatically by the (photographic) optics in photography).

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[76] In general, the ratios (depths) depend (slightly) on the distance (see Par. 2.1.2.1); in any case, the eye can judge the ratio more precisely in case of "identical" objects (see also the longitudinal distortion, Par. 2.1.2.1.1.3.1.1.1).

[77] In reality, information in the memorization means is always coded, but in those cases it is directly decoded by the nuke eye (also if , in the case of holography, to be seen directly by the nuke eye, the initial coding by a "coherent reference field" needs before to be "decoded" by illuminating the hologram with a "coherent reference field" (in certain types, an illuminating incoherent field can "simulate" it).

[78] But in 3D-support memorization, if a correction is not applied during the magnification of the 3D support, the image is distorted at least along the optical axis.

[79] neglecting the de/focusing, so the information that it carries.

[80] It can be recovered, very partially, by using de/focusing degree, and, completely, by the help of the other eye.

Fig. INT.5 – (Natural) perspective is the relation between dimensions of the images on the retina vs. the object distances (M_{transv} variation with the distance of the objects).

[...]

Fig. ... with M_T (Transversal Magnification) that varies as perspective or, better, perspective that derive from M_T (Transversal Magnification) after the focusing process on the retina. Only real images by a positive lens obey to perspective rule (as eye works), magnifier doesn't.

Perspective has a meaning and is useful only when the distances are not recovered in another manner, so only in the mono-objective/monocular case that flattens images by a unique projection on a 2D support[81].

Going "longitudinally" farer from the scene, the angular ratio "/' of the angles subtended by two objects (refer to Fig. *INT.5* in next Par. 2.1.2.2.1.1) at the eye or at the mono-objective diminishes, with a "flattening" effect of the scene since perspective[82] is less efficacious to discover depths when distance increases[83]; because binocular vision reduces to monocular vision for elevated distances, vision is less effective to estimate depths at high distances.

Herein the calculus, seeing/photographing from the *P.o.V.*$|_{3D\ scene}$:

$$' = \tan^{-1}\frac{a}{D_a}$$

---

[81] like a photo or painting support, or the retina.

[82] Perspective is not able to directly measure any distance, so also the "average distance" (the scale factor of the entire scene) (also if the single eye roughly can try). Taking into account this dependence of ratios by absolute distance, ratios of same objects (or of objects of known dimensions) could give, theoretically, information on their absolute distances (or, that's the same, from single angles it would be possible to estimate single distances), but only supposing that objects have the assumed dimensions (and that are not (mignified) models). Practically, we are not able to perceive a longitudinal distortion of a false scene (for example, of a model with foreground objects artificially magnified (and farther) and too far each other, or vice-versa), then, more interesting, of a reproduction of the scene (see Par. 2.1.2.1.1.2.2.1.1).

[83] Reasoning on single angles, calculating the sensitivities of the subtended angle vs. object dimension, *a*, and vs. object distance, *D*: $\frac{\partial}{\partial a}\tan^{-1}\frac{a}{D}=\frac{D}{D^2+a^2}$ and $\frac{\partial}{\partial D}\tan^{-1}\frac{a}{D}=\frac{-a}{x^2+x^4}$ it is clear that the sensitivity is high, with *D* constant, if objects are small, and for objects with equal dimensions the sensitivity increases if *D* diminishes.

$''=\tan^{-1}\frac{c+b}{D_b}-\tan^{-1}\frac{a}{D_a}$, with $c = a \cdot D_b \setminus D_a$ [but in general c must be free]

The perspective is based on the absolute angles or on one distance and the ratio of the subtended angles; here we analyse the ratio:

$$\frac{''}{'}|PoV_{\text{3Dscene}}=\frac{\tan^{-1}\frac{c+b}{D_b}-\tan^{-1}\frac{a}{D_a}}{\tan^{-1}\frac{a}{D_a}}$$

Numerical example:

$a = 40$ cm; $b = 20$ cm; for the particular case of " and ' contiguous, with $D_a = 250$ cm and $D_b = 500$ cm, the “perspective depth effect” is related to the ratio:

"/' = 0.244179

Seeing/photographing the two objects from farer (ΔZ positive), with $\Delta Z = 500$ cm:
$D_a = (D_a + \Delta Z) = 750$ cm and $D_b = (D_b + \Delta Z) = 1000$ cm, the ratio passes to:

"/' = 0.3738435

then, going farther, the two angles " and ' become more similar (with a relative change equal to +53%), so the “perspective depth effect” diminishes.

2.1.2.1.1 Perspective in (2D) drawing/painting and in “virtual reality” (“artificial” perspective)

We can invent a 3D scene, then, by projective transformation, we can apply the “complex” rules of projective to reduce it to a 2D representation, the drawing/painting (loosing scale and depths definitively[84]).

The “evolution” of this method, that is bound to a fixed $P.o.V.|_{3D\ scene}$, is the “virtual reality”, that brings drawing/painting in space domain (and, eventually, also in time domain[85]), and, if desired, in 3D[86].

---

[84] In case of the invention (drawing/painting and “virtual reality”) the distances (or the real dimensions) can be recorded only externally to the representation (as in as in photography, see Note 79).

[85] But not in the domain of dynamics because in virtual reality masses that we can interact with are not present.

[86] We can use one 3D method (for example the simpler one, the two projections method) to represent the invented 3D scene in a more efficacious way.

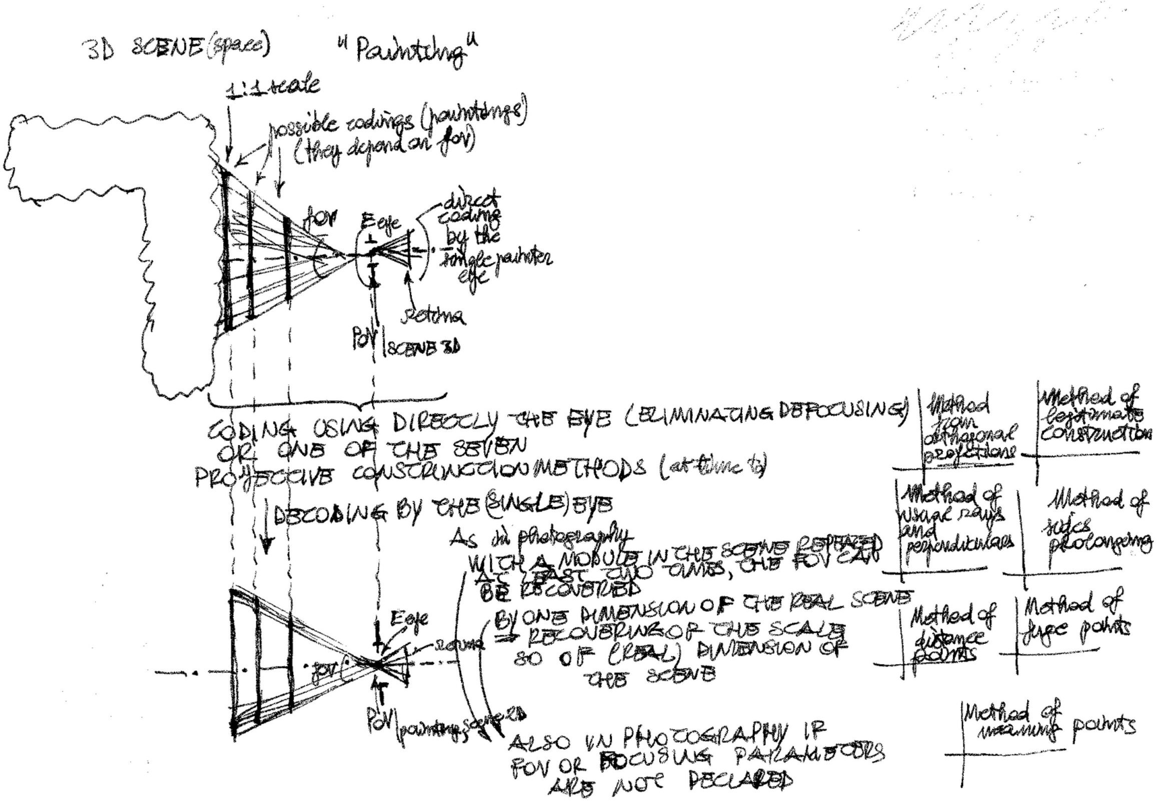
3D SCENE (space)
"Painting"
1:1 scale
possible codings (paintings) (they depend on fov)
fov
Eye
direct coding by the single painter eye
retina
POV | SCENE 3D
CODING USING DIRECTLY THE EYE (ELIMINATING DEFOCUSING) OR ONE OF THE SEVEN PROJECTIVE CONSTRUCTION METHODS (at time t0)
DECODING BY THE (SINGLE) EYE
As in photography
WITH A MODULE IN THE SCENE REPEATED AT LEAST TWO TIMES, THE FOV CAN BE RECOVERED
BY ONE DIMENSION OF THE REAL SCENE
⇒ RECOVERING OF THE SCALE SO OF (REAL) DIMENSION OF THE SCENE
Eye
fov
POV | painting scene 2D
ALSO IN PHOTOGRAPHY IF FOV OR FOCUSING PARAMETERS ARE NOT DECLARED
Method from orthogonal projections
Method of legitimate construction
Method of visual rays and perpendiculars
Method of sides prolonging
Method of distance points
Method of fuge points
Method of measuring points

Fig. INT.6 – Natural and artificial perspective

2.1.2.1.2 Perspective and scale in mono-objective (2D) techniques (photography)

A photographic objective carries out "automatically" a projection, so the ("natural") perspective rules are "automatically" applied and the 2D photographic support can emulate "natural" (monocular) vision.

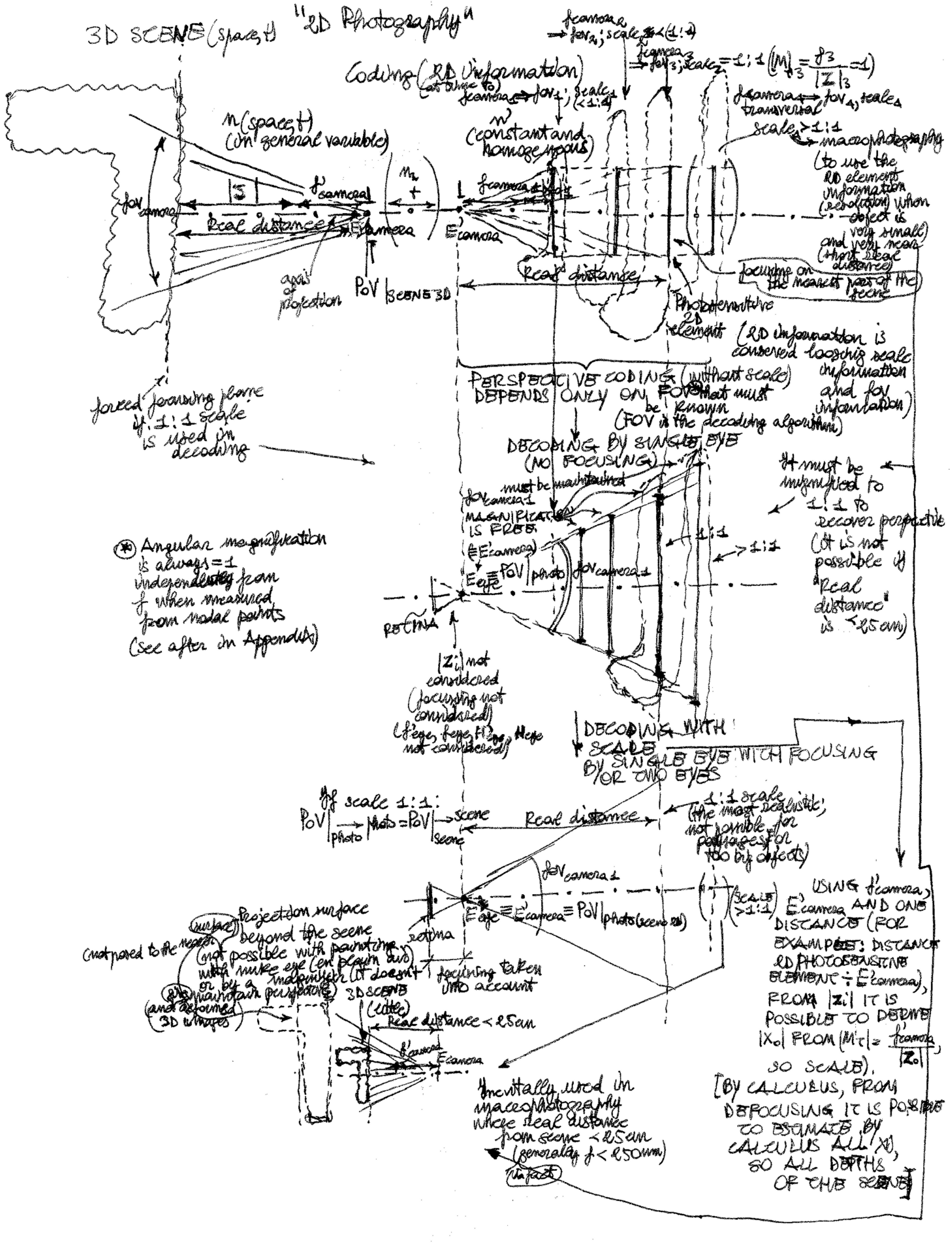
3D SCENE (space, t)
"2D Photography"
Coding (2D information) (at time t0)
f'camera → fov1; scale1 (<1:1)
f'camera2 → fov2; scale2 < (1:1)
f'camera3 → fov3; scale3 = 1:1 (|M|3 = f'3 / |Z|3 = 1)
f'camera4 → fov4, scale4 transversal
scale4 > 1:1 → macrophotography (to use the 2D element information (resolution) when object is very small and very near (short real distance))
n (space, t) (in general variable)
n' (constant and homogeneous)
|S|
f'camera1
f'camera1
fov camera
Real distance
E camera
E' camera
axis of projection
PoV |SCENE 3D
Real distance
focusing on the nearest part of the scene
Photosensitive 2D element
(2D information is conserved loosing scale information and fov information)
PERSPECTIVE CODING (without scale) DEPENDS ONLY ON FOV* that must be known (FOV in the decoding algorithm)
forced focusing plane if 1:1 scale is used in decoding
DECODING BY SINGLE EYE (NO FOCUSING)
must be maintained
fov camera1
MAGNIFICATION IS FREE
(≡ E'camera)
E eye ≡ PoV | photo
fov camera1
1:1
1:1
It must be magnified to 1:1 to recover perspective (it is not possible if "Real distance" is < 25 cm)
RETINA
* Angular magnification is always = 1 independently from f when measured from nodal points (See after in Appendix)
|Zi| not considered (focusing not considered) (f'eye, feye, H'eye, Heye not considered)
DECODING WITH SCALE BY SINGLE EYE WITH FOCUSING OR TWO EYES
If scale 1:1: PoV|photo → photo = PoV|scene → scene
Real distance
1:1 scale (the most realistic; not possible for pathways for too big objects)
fov camera1
E eye ≡ E'camera ≡ PoV | photo (scene 3D)
retina
(scale > 1:1)
focusing taken into account
USING f'camera, E'camera AND ONE DISTANCE (FOR EXAMPLE: DISTANCE 2D PHOTOSENSITIVE ELEMENT ÷ E'camera), FROM |Zi| IT IS POSSIBLE TO DEFINE |X0| FROM |MT| = f'camera / |Z0|, SO SCALE).
[BY CALCULUS, FROM DEFOCUSING IT IS POSSIBLE TO ESTIMATE BY CALCULUS ALL Xi, SO ALL DEPTHS OF THE SCENE]
Projection surface beyond the scene (not possible with painting, with naked eye (in plain air) or by a magnifier (it doesn't maintain perspective))
surface
(not posed to the reader)
gives (and deformed 3D images)
3D SCENE (little)
Real distance < 25 cm
f'camera
E camera
Inevitably used in macrophotography where real distance from scene < 25 cm (generally f < 250 mm)
in fact

Fig. INT.7 - 2D photography method

Photo must be as sharp as possible[87] on the entire *f.o.v.* of the camera, because the photo is not seen with a fixed sight direction[88]. The “out-of-focus”[89] information can not be putted into the photo because the brain uses focusing mechanism on objects at different distances, but all parts of the photo are approximately at the same distance from the eye[90]. So, for quasi-static scenes (really static nature is only an ideal limit), the best photo is a “collage” of many photos taken using a tele-photo[91] zoom objective to cover, with the photographic sensor, a narrow *f.o.v.*[92] to approach the sharp (focused) “linear” (natural) perspective of painting; but during vision the photos must be positioned with different angles[93] and, during the shots, focusing must be carried out by zooming[94] and not by the normal (auto-)focusing system of the camera. In presence of dynamic (not quasi-static) scenes, where the “collage” trick is not possible, a high information content (high # of pixels) in the 2D photosensitive element (as much elevated as it is bigger) permits to make, using a wide-angle[95] lens, a unique photo with an acceptable angular resolution, or to reach a wider *f.o.v.*.

2.1.2.1.2.1 Scene “construction” in mono-objective techniques (and photo-montage, photo-retouch, painting)

Because focusing parameters are not coded into the (mono-objective) photography[96], distance of the scene is not known, so some “tricks” can be used during the photographic shot: 2D objects (greater or) littler than reality can be putted at the right distance in the scene instead of 3D objects (for example 2D representations[97], as 2D photos or 2D paintings[98]) that not forcedly must be to the 1:1 scale of the “real” scene, but realized taking into consideration the $P.o.V.(s)|_{3D\ scene}$ position(s)[99] [100] (see the following Fig. *INT.8*), to obtain no differences respect to “reality”, except for the fidelity[101] [102]. Photo-montage and photo-retouch are made after have taken the photo(/painting), so by calculation (the post-processing), here they are made materially (after calculations) on the scene, with a minor energy efficiency.

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[87] Improving, in the meantime, (2D) fidelity (see Note 16).

[88] eye(s) moves(/e).

[89] “Accommodation information” of the eye, that, in central (foveal) region, always focus the scene.

[90] In any case not at the right distances, because this information is not contained into the perspective projection. One approximation could be a “skew” photo/painting (with the the principal point not in the centre of the photo/painting: a perspective with inclined plane), with the farthest objects in the part of the photo most distant from the principal point (see, Andrea Mantegna, *Lamentation over the Dead Christ*; in any case the plane can not be exactly parallel to the Christ body, or, if it is parallel, it can not respect the distances of the body because of the perspective (projection) definition, that is not sculpture). If the surface of the photo/painting is not flat (planar) (as in the trompes l’œil), the rule must be the same (the most distant objects in the part of the painting most distant from the principal point); in any case it is better that the shape of the surface is not visible by two eyes, and because it is indifferent, the best surface (the most abstract and different from a sculpture of the objects) is the plane (a straight plane). Perspective means to have a whichever surface *here*, instead of a true surface *there*.

[91] Increase of *f* serves to increase the entrance pupil and to decrease mechanical tolerances, with a consequent better angular resolution, but with a proportional decrease in *f.o.v.* (the information capacity of the sensor being constant). Many “cut out” photos with a shorter zoom objective don’t use the full information capacity of the sensor (see Appendix), also if in this less favourable case we could put all photos in the same plane using, during the shots, the multipoint Auto-Focus system of the camera (see later). See Par. *A.1.2.1.1.1* for the choice of focal length vs. $P.o.V.|_{3D\,scene}$.

[92] Theoretically, the narrowest *f.o.v.* as it is possible.

[93] If they are positioned on a plane, the perspective is “widened”, as in certain Canaletto’s paintings (for which he took multiple “photos”).

[94] See *APP.2.1.1.*

[95] Similarly, telephoto lens should be named: “narrow-angle” lens.

[96] This information can be added, but is not contained into the photographic image (see Note 68 for the “artificial perspective”).

[97] Because of their 2D nature, they are surely a representation (necessary but not sufficient condition to be art) of the reality and not the reality itself that is 3D.

[98] background photos/pictures (back-drops) are a special case of them, see Fig. *INT.8*.

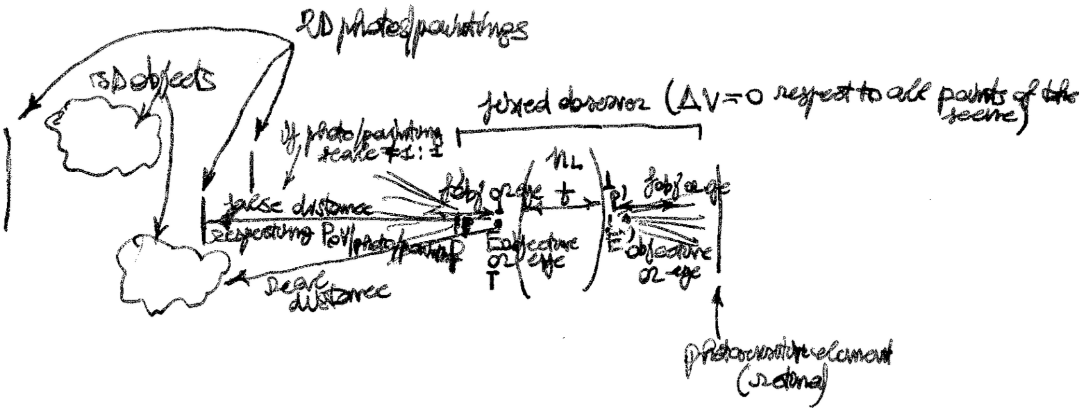
2D photos/paintings
3D objects
fixed observer (ΔV=0 respect to all points of the scene)
if photo/painting scale ≠1:1
false distance
respecting PoV|photo/painting
real distance
nL
E objective or eye
T
objective or eye
photosensitive element (retina)

Fig. INT.8 – Photo of photos

2.1.2.1.2.2 The perspective restitution in 2D techniques

The perspective restitution on the retina, i.e. an accurate reconstruction of the scene starting from the photo/painting, is possible only if the $P.o.V.|_{photo/painting}$ of the observer (the centre of the eye’s entrance pupil) respects[103] the subtended angle to the eye of the painter or to the camera objective (so if it gives the same projective transformation), Fig. *INT.7*. In any case, to well judge distortions amount treated in the following paragraphs, pay attention that in real vision depths and distances of the scene are estimated by stereopsys (binocular vision) and not by perspective/monocular-vision.

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[99] Fixed (in photography) or changing during time (in cinema) $P.o.V.(s)|_{3D\ scene}$. It is necessary to know where the scene will be shot/filmed to correctly decoding the 2D representation. In case of cinema, this is possible positioning, instead of a part of the changing scene/reality, a screen with the 2D or 3D film taken from the same $P.o.V.(s)|_{3D\ scene}$ used to take the film, but only if the $P.o.V.(s)|_{3D\ scene}$ was/were fixed, otherwise the screen should be moved (see Notes 74 and 75).

[100] A photo of a photo can’t be distinguished from a photo of “true” reality (the representation of a representation is a valid representation, also in case of a 3D representations and in general for all representations; in case of a painting of a painting (see R. Magritte, *The human condition*) all remains in the field of invention).

[101] This could deceive also monocular vision, although, in this case, as however in photography, the focusing degree could help to discover the trick, whereas in painting it is impossible (also if “errors” of perspective are present) to discover if it is a representation of a real scene or of an invented scene (both eventually with a distorted perspective) (the trick is perfect because a “reality” external to the painting doesn’t exist, the “reality” is determined by the painting itself, see also the painting by R. Magritte, *The human condition*, cited).

[102] This is not valid for cinema (3D cinema, or also 2D cinema) if there is a relative speed between points of the scene and the camera (see Note 71 and 75): in this case, a 3D “false” scene is necessary (in the Hitchcock's film […] the big cup had to be really constructed) if the camera moves during the ciak or fixed camera and dynamic scene (like futurist painting) → back-drops are not any more valid (neither a 2D screen with a film), but 3D objects must be used.

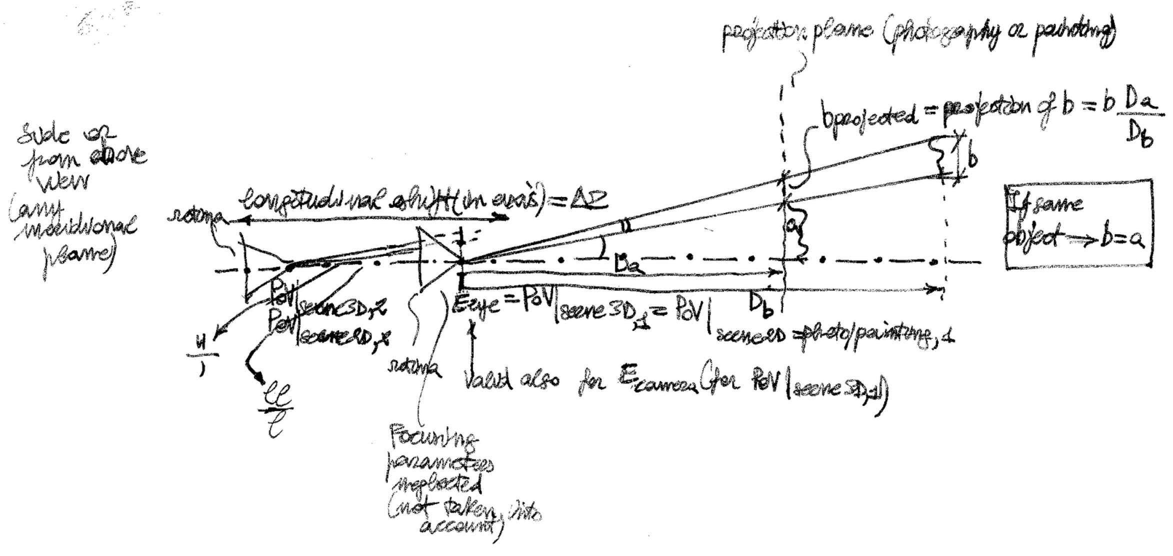
[103] See Par. A.1.3.2 for the pupils position correction.

2.1.2.1.2.2.1 Perspective distortions

The unique[104], correct $P.o.V.|_{photo/painting}$ of 2D representations can be displaced both longitudinally and transversely[105]. Instead of to treat this general case, we split it into its two orthogonal components.

2.1.2.1.2.2.1.1 Longitudinal distortion

The longitudinal "distortion" (squeezing/flattening) is linked to the (estimation) error of the relative distances (the depths), so to the ratio of subtended angles[106]. In the decoding phase, the effect vs. distance is opposite respect to the coding phase treated in Par. 2.1.2.1 (there, going farther from the scene, the "perspective depth effect" was diminished, here it is "anomalously" augmented) because depth information is lost[107] in the (2D) picture/photo, due to a projection according to only one point of view.



projection plane (photography or painting)
side or from above view (any meridional plane)
longitudinal shift (in axis) = ΔZ
retina
retina
bprojected = projection of b = b Da/Db
b
a
Da
Db
If same object → b = a
PoV|scene3D,2
PoV|scene2D,2
Eye = PoV|scene3D,1 = PoV|scene2D = photo/painting,1
Valid also for Ecamera (for PoV|scene3D,1)
H/I
ℓℓ/ℓ
Focusing parameters neglected (not taken into account)

---

[104] Distortions are more complicated in binocular/bi-objective cases, where stereopsys is present, see Par. 2.1.3.1.

[105] Longitudinally: consequent to a shift orthogonal to the focal plane of the projection; in the case of a central projection carried out by an axially symmetrical optical system, along the optical axis. Transversely: consequent to a shift parallel to the focal plane of the projection; in the case of a central projection by an axially symmetrical optical system, orthogonal to the optical axis.

[106] But also the absolute distance of the entire scene and its estimation change with the shift of the $P.o.V.|_{photo/painting}$

[107] without taking into account the out-of-focus information.

Fig. INT.9 - Longitudinal distortion calculation in decoding phase in 2D photo/painting

Referring to the picture above, projecting the object *b* onto the plane of object *a* (putting: $D_b = D_a = D \rightarrow b_{projected} = b \cdot D_a \setminus D_b$) the subtended angles to *a* and $b_{projected}$ from the longitudinally shifted $P.o.V.|_{2D\ scene=\ photo/painting,\ long.\ shift}$ are worth:

$$l|PoV_{scene2D=photo/plainting\,,long.\,shift}=\tan^{-1}\frac{a}{D}$$

$$ll|PoV_{scene2D=photo/plainting\,,long.\,shift}=\tan^{-1}\frac{a+b_{projected}}{D}-\tan^{-1}\frac{a}{D}$$

so the angle ratio, as seen from the shifted $P.o.V.|_{2D\ scene=photo/painting,\ long.shift}$, is worth:

$$\frac{ll}{l}|PoV_{scene2D=photo/plainting\,,long.\,shift}=\frac{\tan^{-1}\frac{a+b_{projected}}{D}-\tan^{-1}\frac{a}{D}}{\tan^{-1}\frac{a}{D}}$$

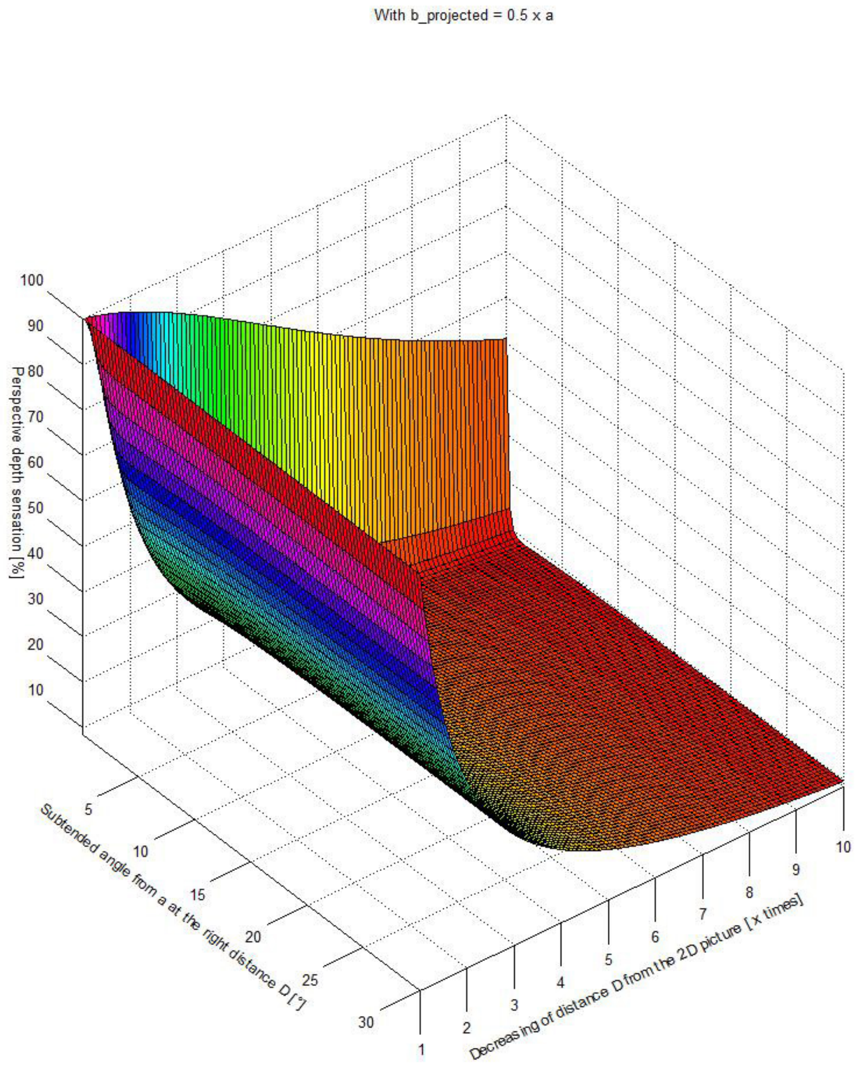


Fig. INT.10 - Longitudinal distortion effect of 2D photo/painting for the particular case: $b_{projected} = a/2$.

Going farther (increasing D), the ratio ll/l increases (for $D \rightarrow \infty$ it tends to $b_{projected}/\ a$), so the estimation by perspective of the relative distances (the depths) is exaggerated (the scene is squeezed and also seen from farther (then it is mignified)); the vice-versa if the distance D diminishes.

Mathematical explanation for this trend:

The trend ("perspective depth effect" increases when D increases) can be foreseen, mathematically, noting that, in the hypothesis of a small angle ratio, the first non-linear term

$\frac{-x^3}{3}$ of the approximation of $\tan^{-1} x$ ($\tan^{-1}x \simeq x - \frac{x^3}{3}$) has the minus sign, so, if D increases, the most important non-linear contribution decreases and the ratio *ll*/*l* increases:

$$\frac{ll}{l} \simeq \frac{\frac{a+b_{projected}}{D} - \frac{(a+b_{projected})^3}{D^3}/3 - \frac{a}{D} + \frac{a^3}{D^3}/3}{\frac{a}{D} - \frac{a^3}{D^3}/3}$$

It brings to:

$$\frac{ll}{l} \simeq \frac{b_{projected} + \frac{-(a+b_{projected})^3 + a^3}{3\mathrm{D}^2}}{a - \frac{a^3}{3\mathrm{D}^2}}$$

As shown in the graph of Fig. 9, this effect becomes evident for big subtended angles[108] (where $\tan^{-1}x$ is very different from x), so in big photos/paintings[109] because in this case the observer can go near to subtend large angles. The effect is also greater with a and $b_{projected}$ not too different one respect to the other (naturally the case: $a = b_{projected}$ is excluded, because the “perspective depth effect”, and consequently its distortion, doesn’t apply).

Numerical example from a practical experiment:

Closing one eye, it is possible to move himself many times away from a 2D picture and to note the longitudinal distortion. For example, with:
a = 12 cm; $b_{projected}$ = 6 cm
moving away of ΔZ = 150 cm starting from D = 50 cm → *ll*/*l* passes from around 0.46705 to around 0.49776, with a squeezing of around 6.6% (or, vice-versa: approaching the 2D picture too much, it seems flattened of 6.6%).

Practical consequences (photography and at cinema)

---

[108] Not too big, otherwise the effect can not be caught by the narrow foveal *f.o.v.* and by the little and rapid rotations of the eye.

[109] So the problem can be present using wide screens (if the spectator goes too near, the picture is less realistic, it is flattened)

In photography, photo enlargement and vision distance are both free, so it is (in theory[110]) feasible to respect, by the spectator, the *f.o.v.* (perspective); besides, the photographer, except for landscapes or too big objects, can also enlarge the photo to have a 1:1 reproduction[111] that is maintained if the perspective is respected.

On the contrary, at cinema the screen dimensions are constant and the position of spectator vs. screen is fixed (he sits), so if the $P.o.V.|_{3D\ scene}$ and/or the zoom factor change during the shot, the perspective is not correctly restituted; with changeable screen dimensions, the *f.o.v.* (*P.o.V.*s) could be maintained, but not together the scale[112]: for both, the spectator should move respect to the screen. At cinema the right distance of about 8-10 metres to the screen is not the usual real distance of objects/persons to the observer/cameraman, so at cinema the reproduction scale on the screen is >1:1 ("iper-realistic"[113]), especially during close-ups[114].

2.1.2.1.2.2.1.2 Transversal distortion

This is the distortion[115] profited by to hide anamorphic pictures. The scene (shapes and depths) are distorted according to the transversal shift (direction and amount).

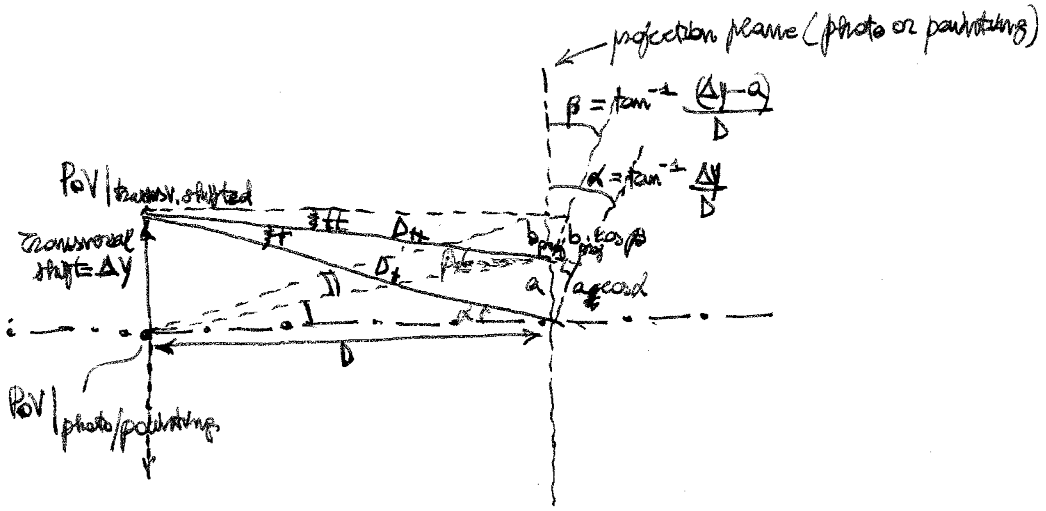

projection plane (photo or painting)
$\beta = \tan^{-1} \frac{(\Delta y - a)}{D}$
$\alpha = \tan^{-1} \frac{\Delta y}{D}$
PoV | transv. shifted
transversal shift = Δy
b_{proj}
$b_{proj} \cos\beta$
D_{tt}
D_t
tt
t
II
I
a
a
D
PoV | photo/painting

Fig. INT.11 - Transversal distortion in 2D photo/painting

---

[110] Who watch the photo normally don't know at which distance to put his eye(s) (usually the photos, for example in a book, are not reproduced with the correct enlargement to be seen from around 25 cm. or there is not a warning to space out, for that photo, the book from his eye(s)).

[111] Referring to the nearer object of the scene (the nearest object to the projection plane, as near as it can touch it).

[112] so the dimensions of the objects on the screen are not correct (the absolute distance of the scene is not the one desired by the movie director). The scale is not maintained also if a fixed focal length objective is used, because the distance $P.o.V.|_{3D\ scene}$/object-in-focus normally changes during the shot.

[113] The distance from the nearest object can be roughly estimated by the spectator that, consequently, can discover that the image of the nearest object is unrealistically magnified (at cinema, all is bigger).

[114] The same also in 3D (bi-objective) cinema.

[115] Differently from longitudinal effects, here not only depths are involved, but, more generally, all the "shape".

Referring to the terminology in the picture above, staying in axis respect to the 2D picture, we can use the results of the longitudinal distortion:

$$'=\tan^{-1}\frac{a}{D}$$

$$''=\tan^{-1}\frac{a+b_{projected}}{D}-\tan^{-1}\frac{a}{D}$$

With a Δy transversal shift:

$$t=\ldots$$
$$tt=\ldots$$

[...]

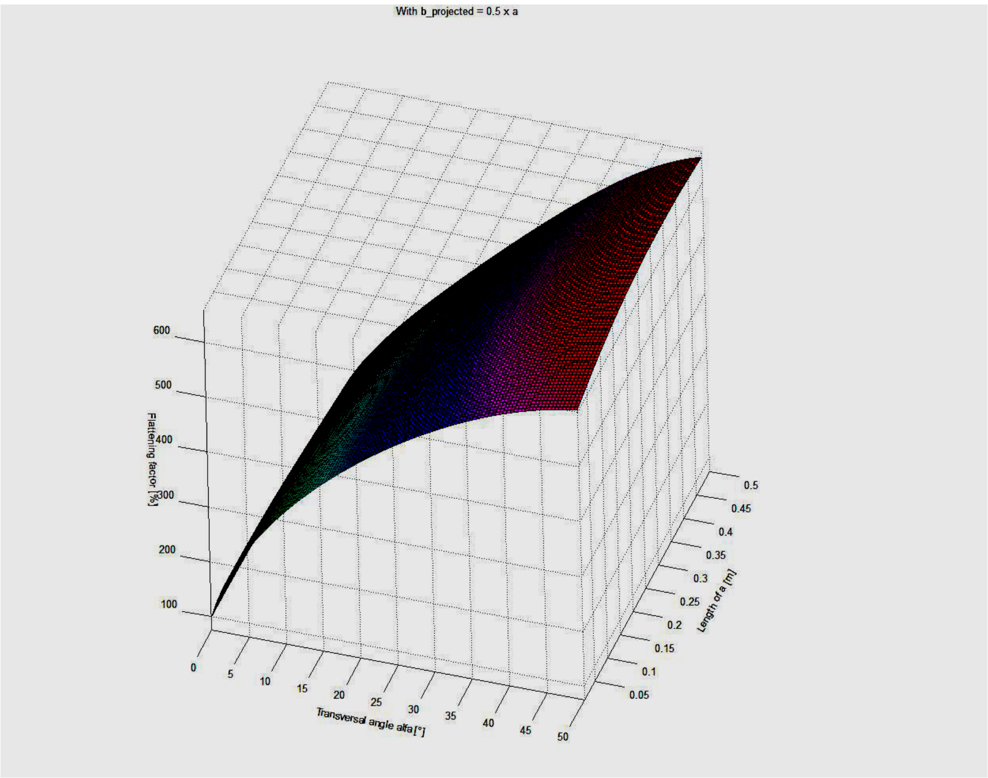


Fig. INT.12 – Transversal distortion effect in 2D photo/painting

2.1.3 Depth determination in binocular and bi-objective photography(/painting[116])

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[116] Theoretically, also a painting/drawing with two calculated projections (perspectives) is possible; the same calculations in real time are at the base of (3D) virtual reality.

Each (nuke) eye optics actuates a projective transformation[117] that projects images onto the retina (that is a (curved) surface, so a 2D geometrical form, i.e. one information (“longitudinal/radial”) is lost[118]). But with the vergence information...

[...]

Fig. INT.13 – (Natural) binocular vision uses the angular displacement.... of the images on the two retinæ to...

[...]

Fig. ... with two eyes

2.1.3.1 Bi-objective photography(/painting) distortions

A coding by a couple of $P.o.V.|_{3D\,scene}$s/photos is more sensible to decoding errors respect to a single $P.o.V.|_{3D\,scene}$/photo as in mono-objective photography; besides, mono-objective image (a single 2D image) is visually an artefact, so distortions can be extrapolated (and eventually corrected), whereas a bi-objective image (the fusion of two 2D images) is less distinguishable by “reality”.

Supposing that the two photos have the same axis (as usual, so they can be superimposed on the same surface/support):

- Longitudinal distortion: the parallel shift along the photos axis of the same quantity provokes a squeezing/flattening of the scene as in the mono-objective case

---

[117] See Note 63.

[118] See Note 64.

- Transversal distortion: the parallel shift orthogonal to the photos axis of the same quantity provokes the same amount of transversal distortion of the scene in each image as in the mono-objective case, but the result (due to stereopsys) is different
- Rotational distortion in space: it is typical of the bi-objective case and causes a complex distortion because left and right images suffer different amount of the two previous distortions, with a complex result in stereopsys. A particular case is the vision of the photo with the neck rotated of ±90° (vision with the horizontal head)[119]: the decoding (fusion) is not possible and the two images are seen[120] separated and vertically shifted
- Intraocular/intraobjective distortion: it is typical of the bi-objective case and causes a complex distortion because left and right images suffer different amount of transversal distortion, with a complex effect on stereopsys .

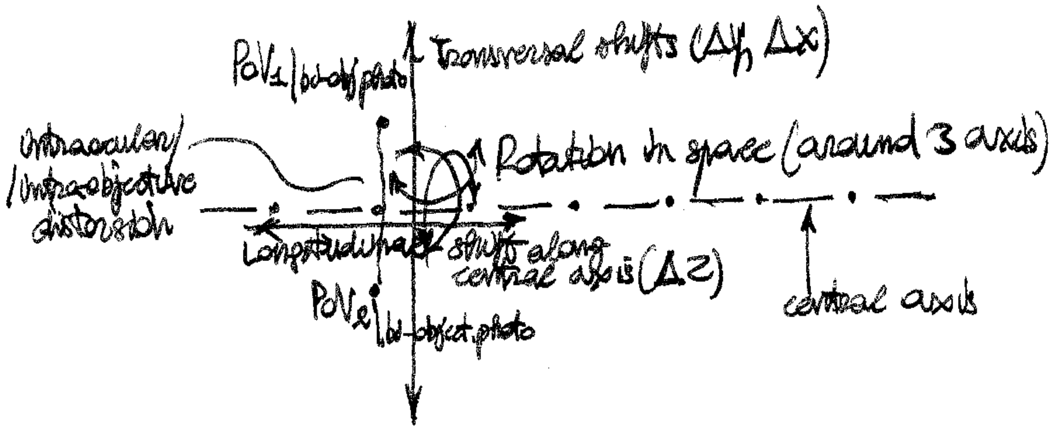

Fig. INT.14 Bi-objective photography/drawing factors that cause distortion. Central axis=axis of the two photos. [to put a 3D object (cube)]

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[119] You can make a test at "3D" cinema or behind a "3D" photo.

[120] If the decoding means (eyeglasses or other) work well after the rotation.

PART A: THE CAMERA AND THE (NUKE) EYE

A.0 The (human) eye and the camera

Both are composed by an optical system and by a mean for detecting and memorising the visible electromagnetic field. A comparison between camera and eye is meaningless for photography performance because the "camera output" (the photo) is subsequently "processed" by the eye: camera must achieve the best performance as possible[121]. Nevertheless, as it will be demonstrated in the following PART C, photography can reproduce "perfectly" the ametropic eye performance when the photo is seen by a corrected/emmetropic eye[122]. In any case, camera and eye can be analysed separately into their macro-components and compared to improve photography and also to cure eye defects[123].

A.1 The optical systems of the eye and of the camera

We start to analyse only the optical part of the eye and of the camera. The optics gives a very little contribution to sensitivity, dynamics and other quantities related to the light *intensity* distribution on the image[124]: a great contribution to them is due to the detector (for an analyse of also the detector, see after in this chapter). On the contrary, it participates, with the detector, to the spatial[125] definition of the image.

A.1.1 Architectures of the optical systems

"Artificial" and "natural" optical systems, as camera and eye, that must *memorise* the image on a support[126], must give a real image on the detector, so must be convergent systems with objects placed in a certain zone, see *APP. 2.1*.

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[121] See previously in the text and Notes 16 and 71.

[122] See Par. 0.3: "The "dream" of the "perfect" photography"

[123] For medical reasons it is necessary to study interactions/parallelism between eye and camera components, as their composition can give back to the defective eye a normal vision or even a better performance. For example, it is possible to use an objective in place of the optical system of the eye to compensate for its poor performance and to apply also a photo-sensor (to arrive to a complete (video-)camera) in presence of a diseased retina (giving back the sight in case of blindness not due to an optical nerve disease). Considering the high performance of photography, we can expect (and hope) that performance of objectives/lenses and sensors are better than those of the human eye. The "substitution" of the optical nerve is not treated here but is another field of research.

[124] For example the image noise that intervenes in sensitivity and dynamics.

[125] Only the spatial, because the timing domain (time information) is (in video-cameras and cinema) mainly defined by detector.

[126] They are systems that can memorise the (electromagnetic energy of the) world, so that permit an *indirect* vision of the world, whereas other optical instruments, as the magnifier, the microscope and the telescope, are *direct* imaging systems: in this last case, the eye directly sees the reality in real time, also if, again, a (video-)camera can be used before, and after the eye (degrading the quality of what is watched, but memorisation is gained: camera+microscope = micro-photography, camera+telescope=astronomical photography).
Magnifier and microscope give virtual images, telescope nor real nor virtual. The paradox of words: a *real* image is for reproduction, a *virtual* image is only for a real time vision.

Comparing camera to the eye, we can try some suggestions to improve photography[127]:

1. Eye moves[128] into its orbit, so it can capture more information than camera, that is not moved when a single instantaneous photo is taken[129]. It is the video-camera for cinema that moves as the eye.

2. A photosensitive element with a hemispherical surface, as the retina, could be introduced to improve aberrations and lighting homogeneity[130] on it. But printing or visualisation of the photo should be made on a not-planar surface...

3. Camera focusing is achieved by moving the fixed-focal-length lens, whereas in the eye the focal length is variable and crystalline eye lens practically doesn't move, so the distance from sensitive layer and optical elements changes (camera) or is fixed (eye). Camera, maintaining the zoom as an additional degree of freedom, should use, to permit a "collage" trick[131] giving the same[132] perspective restitution as if the scene would be seen directly by a collection of eye sights, the eye focusing mechanism (it should use a delta-zooming to focus).

On the other hand, photographic optics can give higher performance vs. eye, because:

- Dimensions (and weight[133]) are not fixed: in general they are bigger than eye ones, also if there are cameras more compact than the eye. Dimensions are a key factor for comparison: as seen in APP.2, optical equations for positions and dimensions of image are invariant to a spatial scale change[134], but, to improve optical spatial quality (see the *MTF*[135]), dimensions must increase (note that the best *MTF* is obtained by tele-photo lenses), nevertheless in this way focusing at very short distances is more impacting[136] (evolution could not furnish a such big eye to man, considering also his need to see at very short distances).

---

[127] We can not improve directly the eye itself that was forged by natural evolution, but we can artificially intervene on it, see Par. A.0 (in this book we will basely limit the analysis to the camera only).

[128] Movement (in particular of the camera in this case) is relied to time domain reproduction, so it is outside the perimetre of the photography. Then there is an asymmetry between eye and camera: photo is taken instantaneously from a fixed $P.o.V.|_{3D\text{-}scene}$, but it is seen during time by an eye that can move in its orbit.

[129] See Par. 2.1.2.1.1.2.1 for the photo "collage".

[130] See A.1.3.6 and there around.

[131] See Par. 2.1.2.1.1.2.1.

[132] In reality, the collage of multi photos taken using an unique pivot (the centre of the entrance pupil of the camera) doesn't restitute the (natural) perspective because the eye doesn't use only one pivot (the centre of the entrance pupil doesn't coincide with the eye or the neck pivots); this error is greater for objects at short distances (respect to the centre of the entrance pupil).

[133] So also complexity, see the following point.

[134] Coming in the eye cure, in ophthalmology the contact lenses actuate the scale change invariant principle respect to the eyeglasses lenses, with a consequent advantage in weight and dimensions, with also an advantage in transmittivity; they have also the advantage to be always centred when eyes are moved.

[135] Diffraction limit improves (consequently *MTF*) with increase in (entrance) pupil dimensions (diameter); and also mechanical construction (tolerances), and aberrations, improve.

- To further decrease aberrations, so to use a higher lens aperture (lower *f#*)[137], cameras have several lenses, instead the eye is formed by only two lenses (cornea and crystalline lens): to space from high to low *f#*, it is necessary to have the degree of freedom of the diaphragm diameter[138], that is possible in the camera thanks to the degrees of freedom of the exposure times and/or of the sensor sensibility[139]. Besides, the camera...

A.1.2 Image fidelity (digression on the general definition of fidelity)

Fidelity of an optical system[140] to reproduce an “original” image[141] is a measure of the lost[142] of information in the reproduction of it, so it depends from two independent domains[143]: temporal-spatial domain (temporal and spatial frequencies[144]) and energy domain (intensities) and can be expressed by sharpness, time[145] fidelity and shade reproduction; considering a monochromatic, static original, respect to the “original”, the reproduction can be very sharp but with uniform intensities, and, on the contrary, the reproduction can have “all” the tones but be unsharp.

Fidelity of the reproduction depends on the system capability to reproduce, of the “original”, its frequencies (temporal and spatial) and their intensities, so it is can be expressed, in general, by the system time-space general response at each intensity and by the time-space intensities of the “original”[146]:

---

[136] In both the focusing methods: with a fixed focal length, due to the elevated delta-shift and big absolute dimensions; with a changing focal length focusing, due to the amount of focal length to be changed; and, for both the two methods, due to the small *d.o.f.*. This is an additional reason, in case of camera, to use the trick of the “collage” (Par. 2.1.2.1.1.2.1).

[137] The increase of the number of lenses permits larger pupil diameters and consequently lower *f#*, but only up to a certain limit: over, losses due to material absorption and Fresnel reflections at interfaces, and weight, become limiting factors.

[138] It must be independent from the environmental luminous conditions (see the next note).

[139] In the eye there are only two sensibilities: photopic (mainly cones) and scotopic (mainly rods), so the diaphragm diameter is not a degree of freedom.

[140] in general of any reproduction system and of any every system

[141] The “original” image is always a reproduction, at least by the eye itself, so its spectral composition (frequencies and intensities) are always *estimated* (see later in the text), otherwise the “original” would be completely known and the perfect reproduction would be possible (see the dream in the Preface, Par 0.3).

[142] So it is a relative measure, as any measure (see the previous Note) (and any statement by physics).

[143] See APP.5.

[144] In general frequencies are used (in Fourier or, more generally, in Laplace analysis where a “complex” frequency is used, i.e. a real frequency f_t and a phase θ).

[145] See the following Note 132.

[146] We suppose that system (=observer) and scene have not a high relative acceleration, because in this case time and space of the system and of the scene are not the same, and timespaces of the system and of the scene are not yet the same (subjected to the restricted theory of relativity), but time and space are different (we have time and space of the scene as added unknowns).

Image Fidelity(System response ((f_t , θ_t), (f_{s1} , f_{s2} , $\Phi_{s1,\ s2}$))[147]); intensities); “original”[148] (frequency composition; intensities))

According to the relativity, (f_t , θ) and (f_{s1} , f_{s2}, $\Phi_{s1,\ s2}$) domains are related[149] by the phase speed (*c* in free-space or *n* in a medium)[150].

In our research of a total image fidelity function, we consider separately the time and the spatial domain (two multiplied independent functions):

Image Fidelity(System response (f_t , θ_t) x System response (f_{s1} , f_{s2} , $\Phi_{s1,\ s2}$)); intensities); “original”((temporal composition; spatial composition); intensities))

In the hypothesis of a system able to perfectly reproduce the intensities of whatever frequency (so that it is independent from the intensities) (it is a *linear* system), its response is simplified by its responses to a *unitary* time and spatial pulse:

System response ((f_t , θ_t) , (f_{s1} , f_{s2} , $\Phi_{s1,\ s2}$)) UGUALE CIRCA PROPORZIONALE [|F[*Pulse*] (f_t , θ_t) | x | F[Fraunhofer *PSF*| (f_{s1} , f_{s2} , $\Phi_{s1,\ s2}$)] |]

so, considering static and monochromatic[151] 2D scenes, it can be expressed by the *MTF*(f_{s1} , f_{s2})[152] (that is expressed for a constant (normalised) intensity):

System response(f_{s1} , f_{s2}) UGUALE CIRCA *MTF* = | F[Fraunhofer *PSF* (f_{s1} , f_{s2})]|

For a general system (a non-linear system), for example by the *MTF* and by the dynamical range (linearity function[153]) that, in general, is function of spatial frequencies:

System response((f_{s1} , f_{s2}); intensities): [*MTF; dynamical range* (f_{s1} , f_{s2})]

---

[147] Three variables, in general constituted by the phase and two spatial frequencies; for 2D images only the two spatial frequencies are used.

[148] Estimated, see the previous Note.

[149] According to the special theory of relativity (applied to optics through Maxwell equations).

[150] Time frequencies have influence on spatial fidelity through the phase speed change (the change of index of refraction vs. frequency = dispersion), bringing to the chromatic aberration.

[151] To avoid time treatment: static to avoid the time transients, monochromatic to avoid time frequencies. Transient reproduction will not be treated, frequency reproduction (in time (=colour reproduction) and in space chromatic aberration) will be treated. The term “dispersion” could be used not only relating to the degradation of the spatial reproduction of different colours (for the time spectrum enlargement and its influence on the space through the speed), but also for the sharpness worsening in space (enlargement of the spatial spectrum).

[152] Because we consider only incoherent systems, in the spatial domain we use only the module of the *OTF*, said *MTF* , so spatial frequency distortion can be only due to amplitude distortion and not phase distortion.

[153] Non-linear, in reality.

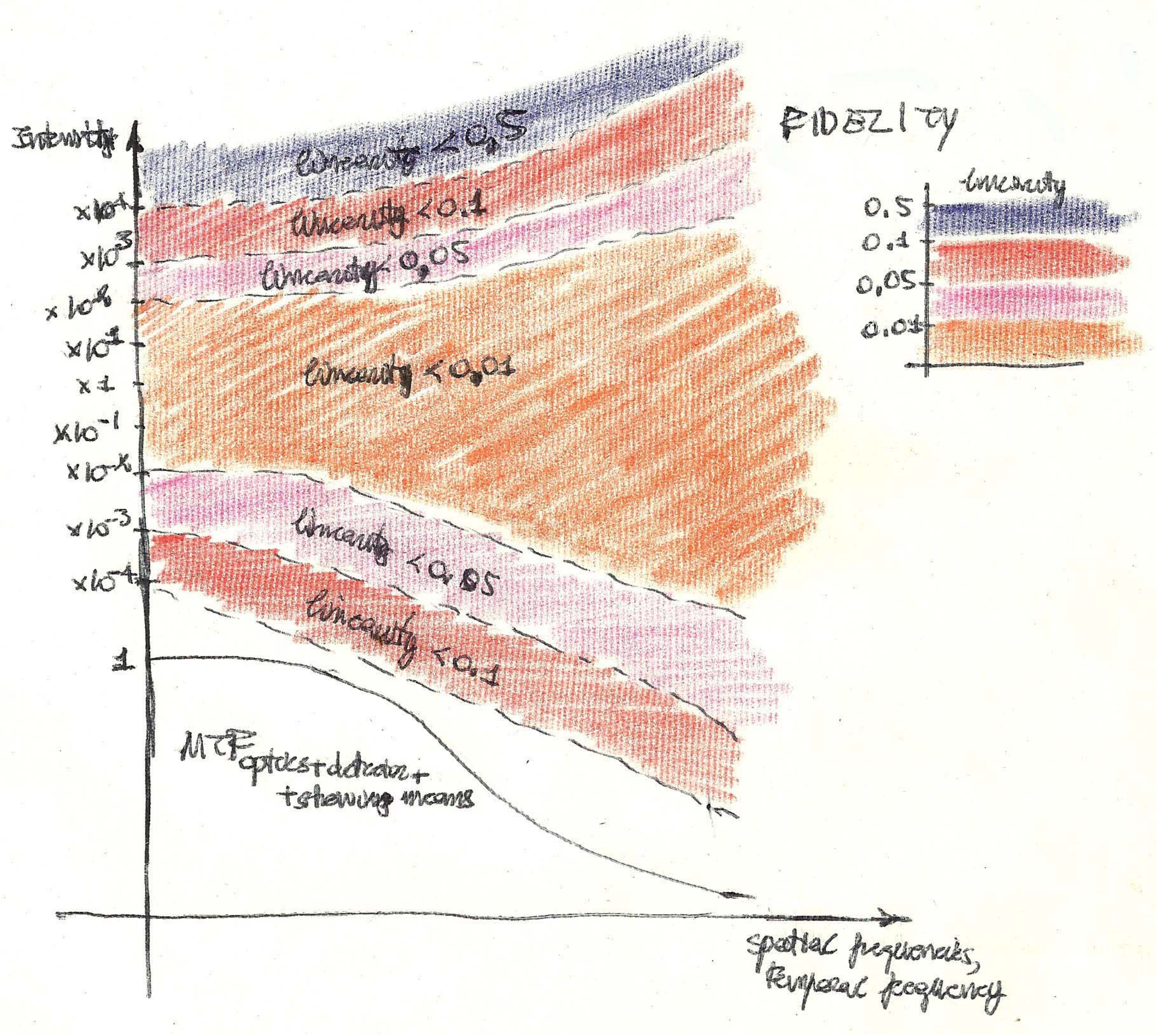

Fig. A.1 – The two "Fidelity" functions [temporal frequencies to be cancelled]

MTF(f_{s1} , f_{s2}) is defined both by optics[154] and by detector/visualisation-means; linearity, instead, is practically function of only detector+visualisation-means (optics is quasi-perfectly linear).

For fidelity evaluation of the spatial frequencies reproduction see the following paragraphs.

For (incorrigible) errors (depending on the light intensity) that impact linearity and dynamical range (noise comprised towards low), see after in this Part A; also some (energetic) uniform[155] errors, neither involving linearity nor dynamical range (so corrigible errors during post-processing if the image intensity is inside the linearity range), like Attenuation, Aperture

---

[154] For optics, because generally it has axial symmetry, it has the same symmetry, so it can be function of only one f_s.

[155] On the entire intensity (DC factor) as attenuation, or on a part of the image (Aperture Efficiency, Vignetting, Relative Illumination). They can be due to optics or also to detector or reproducing means (those are corrigible too).

Efficiency, Vignetting, Relative Illumination, are considered. To resolve the dynamical range limitation of the detector (not of the visualisation means) and to use it in a range the most linear as possible, more[156] photos of the same scene can be taken at different apertures, then collected together biasing each one with the correct relative intensity: so the previously analysed collage trick for the best focusing can be extended[157] to a collage trick for the best "intensity fidelity", too.

A.1.2.1 (Spatial) frequency response of the optics

In this paragraph we consider the contribution of optics to the (spatial) sharpness. We start with the evaluation using geometrical optics, and immediately after we'll refine it using wave optics. In both the analysis we suppose that (spatial) response is independent from the intensities, then a response to a single (unitary) punctiform source is independent from the responses to the other points, so sharpness can be treated through focusing analysis.

A.1.2.1.1 Focusing (geometrical optics is insufficient[158])

As a general principle, in any physical/natural process (and also in the act of reproduction, hypothesising that we have free will), a part of both temporal and spatial information is loosen[159], then both pulse response and point spread function are not ideal (not a temporal pulse or a spatial point are sharply reproduced). In particular, remaining, artificially, only in the spatial domain, a perfect focusing would mean do not loose spatial information, so to conserve the spatial resolution.

For in-focus objects, in the ideal paraxial case (perfect imaging, i.e. geometrical optics without aberrations) the blur spot is zero[160]. More generally, in geometrical optics we introduce a no-null "circle of less confusion" that determines the best focus position in case of the omnipresent aberrations:

......δx……… → circle of less confusion =....

To measure the out-of-focus amount for out-of-focus points, an acceptable "blur circle"[161] (here: "blur spot", because, in general, it is not a circle) is introduced, that in wave optics will be generalised by the Fresnel Point Spread Function[162].

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[156] Theoretically, the number of photos should be infinite to use the infinitesimal range of linearity of the detector (of the system).

[157] As in the other collage, also this one can be done if the scene is quasi-static. Also in this case, the trick is simpler with digital photography (with post-processing techniques).

[158] For a more precise discussion following wave optics, see Par. *A.1.2.2.1.*

[159] As also on intensity fidelity (and total intensity due to losses).

[160] In wave optics the zero blur spot will be generalised by the (Fraunhofer)Point Spread Function (see Par. *A.1.2.2.1*): the point is spread, its image is never punctiform.

[161] Circle Of Confusion is used in other texts, also if, practically, it is never a circle.

[162] See Par. *A.1.2.2.1.*

In the following, in particular we will consider the “fixed”[163]refractive “defects” of the eye; they are completely covered by myopia and hyperopia[164]. Both of them give the same perceived blur effect[165], but it differently depends on the object distance: the myopic eye sees has a shorter *d.o.f.* for the same distance and can not focus at long distances, but can arrive to focus down to the myopic near point, *nearer* than emmetropic eye near point; on the contrary, hypermetropic eye can see as the emmetropic eye at infinity and decreases its performance as the distance decreases. To try to improve focusing, a short-sighted person “should relax” as much as he can his crystalline lens, on the contrary the far-sighted person gets tired to accommodate it.

A.1.2.1.1.1 The longitudinal magnification and the d.o.f. and their relations with the blur spot (geometrical optics is not enough)

Ideally, the extension of the depth of field of the camera should be infinite, to have all the entire scene in-focus[166]. In optical systems with projection on a surface and not in a volume (as in the eye and in the “traditional photography”, see Introduction, Par. *0.1*) has no sense to speak about focusing, but it would be more correct to speak about the amount of defocus, because (“perfect”[167]) focusing of the scene is possible only if it is constituted by just one single point, so an optical system with *f.o.v.* = 0° should be used, or objects should be exactly at infinity, to have $M_{long} = 0$[168], whereas camera objectives (such as the eye optics) mignify but not down to zero ($M_{long} \neq 0$), so the image has a longitudinal dimension and is not (perfectly) focused on the surface of the detector. For this reason, a not-infinite extension of *d.o.f.* must be accepted, fixing an “acceptable linear delta[169] blur spot”, $\Delta b'_{blur,acceptable}$ on the retina, when we directly analyse the image, or also an “acceptable delta blur angle”, $\Delta\beta_{blur, acceptable}$ expecially useful when we are watching to a photo (because we want to respect perspective restitution, *d.o.f.*)[170] (refer to the following Fig. *A.2*).

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[163] The “variable” focusing defect, i.e. the defect in the focusing process, is the presbyopia.

[164] See Par. *A.1.2.2.1* for a wave optics treatment of them.

[165] See later and in the Appendix for the confirmation in the wave optics approach: except that when in myopic eye the object is at the myopic near point (so in-focus), the *Fresnel MTF* and *Fresnel Point Spread* functions do not depend on the sign of the dioptres.

[166] The *d.o.f.* must be large enough also for shots of objects having an high longitudinal speed: in photo-cameras, if the illumination is high enough, an insufficient *d.o.f.* can be compensated by using very fast shot times, whereas in video-cameras the auto-focus can not be fast enough, so the need for a large *d.o.f.* for rapid changing scenes is more critical (video-cameras must film rapid objects from faraway).

[167] again, only in geometrical optics and in perfect imaging hypothesis.

[168] See the M_{long} formula in Appendix (at infinity objects are focused in *F'*). Nevertheless, for objects much farer, from the front principal point, than focal length, *f*, the focusing is very good.

[169] an approximated delta (according to geometrical optics) respect to the initial blur spot (due to diffraction and aberrations), so an “amount of out-of-“focus” ”. We use here a delta blur and not an absolute blur, that can be the key factor (influencing *d.o.f.*, near emmetropic point and ametropic points, see later). It can be calculated knowing contribution by optics: aberrations (and adding diffraction), and, for the detector contribution (retina or photographic means) their blur spots, that do not depend on the “optical parameters” *E* and $s_{focused}$.

[170] Because we maintain the perspective, so with the entrance pupil of the eye in the good position, $\Delta\beta_{blur,\ accept.}$ at the eye is equal to $\Delta\beta'_{blur,\ accept.}$ of the photography that, so, must be equal to the $\Delta\beta'_{blur,\ accept.}$ of the eye. $\Delta b'_{blur,accept.}$ and $\Delta\beta'_{blur,\ accept.}$ must have values of the same order of $b'_{blur,accept.}$ and $\beta'_{blur,\ accept.}$ (smaller to take into account for the

Referring to Fig. *A.2*, it would be necessary to express the initial blur spot (due to the pure diffraction and to aberrations ("aberration spot")) as a function of entrance pupil E, and $s_{focused}$, but, to avoid it, we *approximate* that this "constant"[171] initial blur spot can be added, at the end, to $s'_{d.o.f.,\,front}$ and $s'_{d.o.f.,\,rear}$.

Applying a proportion[172] in the image side, then transferring it in the object side:

Front d.o.f. / Rear d.o.f.[173] [m] $= s_{focused} - s_{d.o.f.,\,front}$ / $s_{d.o.f.,\,rear} - s_{focused}$ (positive[174] quantities)

We can derive two different forms:

1. For not too high changes of the entrance pupil E, we can use this general approximated[175] formula (that uses the *linear* quantity $\Delta b'_{blur,accept}$):

$$\textit{Front d.o.f. / Rear d.o.f.} = \textit{CIRCA}\ \frac{\Delta b'_{blur,\,accept}(s_{focused}-f)[(P_M-1)(\pm s_{focused}\mp f)\pm s_{focused}]}{\Delta b'_{blur,\,accept} P_M (s_{focused}-f)\pm f\,P_M E} \qquad (1)$$

This form is especially used for the analysis of the **eye** itself, with[176]:

$f = f_{Eye}$; $P_M = P_{M,Eye}$; $E = E_{eye}$

In the eye, f_{Eye} and $P_{M,Eye}$ change with accomodation, so with $s_{focused}$ (see the following numerical example for the eye and see also Fig.*APP-24*).

For $s_{focused} > s_{focused,\ hyperfocal} = \dfrac{f\,P_M E}{\Delta b'_{blur,\,accept} P_M} + f$, the *Rear d.o.f.* denominator is null and the *Rear d.o.f.* formula is not valid any more (it is $+\infty$ m).

"initial blur"), values ("sharpness amount") that depend on the *complete eye* (optics + retina + brain) resolving power: $b'_{blur,accept.}$ of the order of cone separation (3 μm), $\beta_{blur,\ accept.}$ of the order of 1' (see later the numerical example on the camera) ($\Delta\beta_{blur,\ accept.}$ $\Delta\beta blur,\ accept.$ should be theoretically zero to have, in the photo, the same degree of focus in all the foveal range of who watch the photo; nevertheless, the neighbourhood can be better in focus than during direct eye vision (the *relative d.o.f.* of a reproduction can be better than reality!: the photo has, forcedly, a $\beta_{blur,\ accept.}$ worse everywhere, but can be more uniform than direct vision: for example, a photo that is able roughly to focus both objects very far one respect the other: a stick on a glass at 50 cm and another object in the far background, both of them that can not be focused at the same time by the *fixed* eye) (see also Note 13)); normally a $\Delta\beta_{blur,\ accept.}$ equivalent to $\Delta b'_{blur,accept.,\ Eye}$ is chosen for watching to a photo). In the following, we will calculate explicitly, for photography, only the delta contribution of optics (not of support/showing means).

[171] vs. $s_{focused}$ and, at a less extent, vs. E.

[172] So *Front d.o.f.* and *Rear d.o.f.* don't depend on n'/n. But in the following formula with angles, n'/n will be present because when M_{ang} is unitary, at centres of the entrance and exit pupils (the *f.o.v.* calculated from them is maintained, so there perspective is respected), it is linked to the linear transversal magnification M_{transv} by n'/n (see APP.2: $M_{ang}|_{z\text{-}coordinate=constant} \cdot M_{transv}|_{z\text{-}coordinate=constant} = f/f' = n/n'$) that is not unitary for the eye.

[173] Always *Rear d.o.f.* > *Front d.o.f.* because in convergent systems in the analysed range $s_{focused} > s_{d.o.f.,\,front} > f$ (see also Note 93) M_{long} increases for a decreasing s. *Front d.o.f.* + *Rear d.o.f.* = *d.o.f.* , but here we conserve the distinction.

[174] If *Rear d.o.f.* is negative, $s_{focused} > s_{focused,\ hyperfocal}$ so *Rear d.o.f.* formula is not any more valid.

[175] approximated also because we use a constant $\Delta b'_{blur,accept}$, otherwise it should change taking into account for the "initial blur" changes with E and $s_{focused}$

[176] Except for $n_{vitreous}$, the other quantities must be calculated at the distance $s_{focused}$ that is not infinity (uncommodated) because at infinity *d.o.f.* has not sense.

An instructive particular case of the general formula is when P_M is fixed and equal to 1[177]:

$$\textit{Front d.o.f. / Rear d.o.f.} = \frac{\pm \Delta b'_{blur,\,accept}\left(s_{focused}-f\right)s_{focused}}{\Delta b'_{blur,\,accept}\left(s_{focused}-f\right)\pm f\,E}$$ that depends on E, f and $s_{focused}$ also if it doesn't depend directly on $f\# = f/E$

2. For not too high changes of the entrance pupil *E,* we can derive this second form, from the previous formula: it uses the *angular* quantity $\Delta\beta_{blur,\ accept.}$, following a supplementary approximation[178]:

$$\textit{Front d.o.f. / Rear d.o.f.} = \frac{\Delta\beta'_{blur,\,accept}\left[\pm\left(s_{focused}-f\right)^2\left[\left(n'/n\right)\left(P_M-1\right)^2\right]\pm\left(s_{focused}-f\right)\left[2\left(n'/n\right)\left(P_M-1\right)s_{focused}\right]\pm\left(n'/n\right)s^2_{focused}\right]}{\Delta\beta'_{blur,\,accept}\left[\left(n'/n\right)\left(P_M-1\right)\left(s_{focused}-f\right)+\left(n'/n\right)s_{focused}\right]P_M\pm P_M\,E}$$

(2)

This form is useful for the analysis of the **photo**.

Due to the increasing of M_{long}[179], when $s'_{d.o.f.,\ rear} = s'_{d.o.f.,\ rear,\ min} = f'$, $s_{focused} > s_{focused,\ hyperfocal}$.

For $s_{focused} > s_{focused,\ hyperfocal} = \dfrac{\dfrac{E}{\Delta\beta'_{blur,\,accept}}+\left(n'/n\right)f\left(P_M-1\right)}{\left(n'/n\right)P_M}$ (when the *Rear d.o.f.* denominator is null) the *Rear d.o.f.* formula is not valid any more (it is +∞ m).

The instructive particular case with $n' = n = 1$ and P_M fixed and equal to 1, worths:

$$\textit{Front d.o.f./ Rear d.o.f.} = \frac{\pm\Delta\beta'_{blur,\,accept}\,s^2_{focused}}{\Delta\beta'_{blur,\,accept}\,s_{focused}\pm E}$$

It doesn't depend on f (here f_{camera}) and shows that, for a certain $s_{focused}$, if the camera optics has an high and constant[180] exit pupil equal to the entrance pupil (so $E=E'$ is fixed and high) its *"relative" d.o.f.* is poor[181].

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[177] For the eye it is an impossible case because the eye pupil magnification is variable with accomodation and different from 1 (See *APP.2.2.1*).

[178] For small $\Delta\beta_{blur,\ accept.}$, then we must introduce an approximation to the approximated formula above; considering that $\beta_{blur,\ accept.}$ is small, we can substitute the linearised expression for $\Delta b'_{blur,accept.}$ (see Fig. *A.2*) in the above formula.

[179] See Note 90.

[180] So we are analysing, as in the following numerical example, a fixed focal lenght objective.

[181] From the following numerical example, it will be clear that to arrive to the good *d.o.f.* of the eye at the same distance ($s_{focused}$) (to maintain perspective), an objective with an absolute E similar to the little absolute E_{eye} must be chosen.

Numerical examples for the eye and for the camera

Eye (emmetropic eye):

With $\Delta b'_{blur,accept.} = 10\ \mu m$ and $E = E_{eye} = 2.5$ mm (good ambient/object luminosity):

For a distant object (it is for 4.187 m) → the eye is around uncommodated →[182] $f =\approx f_{Eye, uncommodated} = +16.68196$ mm; $P_M = P_{M,Eye, uncommodated} = 0.92 \rightarrow s_{focused, hyperfocal, Eye} = 4.187$ m.

For a distant object, but $< s_{focused, hyperfocal, Eye}$: $s_{focused} = 4$ m → the eye is around uncommodated →[120] $f =\approx f_{Eye, uncommodated} = +16.68196$ mm; $P_M = P_{M,Eye, uncommodated} = 0.92$:

Front d.o.f.$_{Eye}$ / *Rear d.o.f.*$_{Eye}$ = 1.955 m / 85.157 m

For a very near object (fully accomodated eye) we can calculate the emmetropic near point): $s_{focused, fully\ accomodated, emmetropic\ Eye} = 0.1434$ m → [183] $f_{Eye, fully\ accommodated} = +14.7742$ mm; $P_M = P_{M,Eye, fully\ accommodated} = 0.8476$:

Front d.o.f.$_{Eye}$ / *Rear d.o.f.*$_{Eye}$ = 4.9151 mm / 5.2698 mm

So, with the chosen $\Delta b'_{blur,accept.} = 10\ \mu m$ and $E_{eye} = 2.5$ mm, for the LeGrand fully accomodated emmetropic eye the nearer point that can be acceptably focused is at:

$s_{focused, fully\ accomodated, emmetropic\ Eye}$ - *Front d.o.f.* - 1.8186 mm = 143.4 mm - 4.9151 mm - 1.8186 mm = 136.6663 mm from the first vertex of the cornea.

Nevertheless this very short distance, $|M_{transv}| = 0.11486$ is not very high[184] because f_{Eye} is very short.

If you see with only one (nuke) eye, and focus on a near object, farther objects (in the background) and nearest objects (in the foreground) are not perfectly focused (and in a no-symmetric manner[185]: nearer objects are worse; see Fig. *A.2bis* for the best point to focus to have the best *d.o.f.* according to the depth of the 3D scene). But in general, when the object is not too near, the weakness of the effect is due to the fact that the *d.o.f.* of the emmetropic[186] eye is elevated[187] and almost equal to the total emmetropic focusing range (that goes from the emmetropic minimum focusing distance, $s_{focused,\ near\ point,\ emmetropic\ Eye}$, to infinity), for the following reasons:

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[182] See Appendix.

[183] See Appendix.

[184] We are usual to compare these values to the camera ones, but the eye should be our first comparison criterion.

[185] See the longitudinal magnification formula in the *APPENDIX* and its asymmetry there discussed (see Note 115).

[186] For a myopic or an hypermetropic eye, the *d.o.f.* is less high (see after the *Ametropic Eye objective* section, and Fig. *A.3).* In particular, a paint of a myopic painter has near objects in focus and far objects out of focus, as the “romantic” effect of photography playing on short *d.o.f.*.

- f_{Eye} is short
- E is little, thanks to the compactness of the eyeMETTERE O GUARDARE FORMULA CON Mt e Ml, so $f\#_{Eye}$ is high (it can be high despite short f_{Eye} thanks to the high sensitivity of the retina and to the fact that a not very fast[188] refresh time is demanded)

(On-focus) Camera[189]:

The photo is supposed to be well magnified, so to be watched by the emmetropic nuke eye from 1 metre (see Note 87). Adopting the same acceptable blur criterion of the eye ($\Delta b'_{blur,accept.}$), we calculate the $\Delta\beta'_{blur,accept.}$ from the eye geometry[190]:

$$\Delta\beta'_{blur,\,accept} = \approx \frac{\Delta b_{blur,\,accept}}{(n'/n)\,f\left[\frac{s_{focused}}{s_{focused}-f}+(P_M-1)\right]}$$; with $\Delta b'_{blur,accept.}$ = 10 μm, $f = f_{Eye}$ (1 m) $=\approx f_{Eye}(\infty\ \text{m}) = f_{Eye,\ uncommodated}$ = +16.68196 mm; $P_M = P_{M,Eye}$ (1 m) $=\approx P_{M,Eye,\ uncommodated}$ = 0.92; $n' = n'_{vitreous}$ = 1.336; $s_{focused} =\approx \infty \rightarrow \Delta\beta'_{blur,accept.}$ = 1.6766 '

We don't decrease this value to take into account for the reproducing means "granularity" because at 1 metre it is negligible.

For a distant object < $s_{focused,\ hyperfocal,\ camera}$: $s_{focused}$ = 4 m (as in the above example for the eye, to be in the same *P.o.V.* to conserve perspective[191]). With fixed[192] objectives of f = +50 mm and f = +30 mm and $f\#$ = 8 for both these two cases (so: E = 6.25 mm and E = 3.75 mm, greater than the E_{Eye} above), n' = 1; P_M = 1 (typical for a standard objective of f = +50mm) and $\Delta\beta'_{blur,accept.}$ = 1.6766 ' (= 4.876 10^{-4} rad):

Front d.o.f.$_{camera}$ / *Rear d.o.f.*$_{camera}$ = 0.951 m / 1.8145 m and 1.368 m / 4.335 m

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187 Expecially vs. long camera objectives (see following numerical examples); the drawback is that the eye sharpness is not very high due to the small entrance pupil diameter.

[188] Time information (time definition) is not as good as a camera when camera uses fast exposure times. The time *x* space information product is the real global information quantity of interest and is proportional to the sensor sensitivity, to the sensor's spatial and temporal resolution, and to the *aperture diameter* so to the *f#.* In the eye, sensor temporal resolution (because it generates a film as a video-camera, so continuous images, whereas the camera capture a single frame, so its time-space product is instantaneous and not continuous [it doesn't give a channel capacity]) and eye aperture diameter are not so good as a camera. Camera, playing on its "one shot" temporal resolution, can give pictures that can not be detected (seen) by the eye.

[189] With a fixed focal length objective.

[190] That sligthly depends on f_{Eye} so on the distance of the photo from the eye. $\Delta\beta'_{blur,accept.}$is maximum for maximum $f_{Eye,}$ ($= f_{Eye,\ uncommodated}$ = +16.68196 mm, see APP.2.2.1).

[191] In reality, we are using $s_{focused}$ instead of distances from E, so with an approximation on the distances of the objects, expecially for short distances.

[192] We are considering fixed focal lengths vs. $s_{focused}$, so we are not considering the collage of multi-photos with focusing by changing the focal length, to obtain a not-approximated perspective (with the trick of multi-photos of narrow *f.o.v.* taken with variable focal length, we correct the standard artificial and mono-photo perspective and we arrive to "imitate" the natural perspective of the eye (that moves *H, H', F, F'* for v.s each focused point).

For a near object: $s_{focused}$ = 0.1434 m[193] (as in the above example for the eye). With a macro[194] wideangle lens of f = +28 mm and f = +15 mm and $f\#$ = 8 for both the cases (so: E = 3.5 mm and E = 1.875 mm, similar to the E_{Eye} above), n' = 1; P_M = 1.5 (typical[195] for a macro objective) and $\Delta\beta'_{blur,accept.}$ = 1.6766' (= 4.876 10^{-4} rad):

Front d.o.f.$_{camera}$ / *Rear d.o.f.*$_{camera}$ = 3.6537 mm / 3.86429 mm and 7.089 mm / 7.8982 mm

It is evident that normally (with usual E) camera has, respect to the eye, a reduced but acceptable *d.o.f.* at long distances, whereas at short distances its *d.o.f.* is so short that it is difficult to focus any entire scene: at short distances, especially for dynamic scenes, the direct eye vision is irreplaceable because it has a similar *d,o,f,* but it moves to focus all the neighbourhoods of the focused point. To improve at maximum *delta d.o.f.* of one single photography it would be convenient to take the photo from as far as possible (with a E as large as possible), but this could not respect the chosen $P.o.V.|_{3D\ scene}$ and the spatial resolution of the detector could be not enough. The best way is the following one (with or without the collage[196] method): given the $P.o.V.|_{3D\ scene}$, we calculate the maximum f (knowing that $f < s_{d.o.f.,front} < s_{focused}$), so we can increase the E as much as is possible (with the limit of $f\#$, because a too low $f\#$ improve luminosity but causes aberrations that worsen the “initial” blur spot that is improved by the increasing of E (less diffraction)).

With the camera:

1. The main goal is the maximum sharpness: for (quasi-)static scenes the trick of multiple photos is useful[197], using a telephoto lens and a tripod to move the camera around the centre of its entrance pupil as a fulcrum. But, even if the object is a 2D one, it is not perfectly on focus, because its non-null transversal extension brings to a change in object distance respect to the photosensitive element (also in the case of using a strange hemispherical photosensitive element, if the object surface shape is different from a hemispherical one).
2. On the contrary, a “Romantic effect” (not present in eye vision and in painting[198]) is possible: some objects can be out-of-focus, playing on the short *d.o.f.* of some long and fast objectives. In this case, the photo(grapher) determines not only the $P.o.V.|_{3D\ scene}$, but also the direction of sight of the spectator.

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[193] Camera vs. Eye: the near point of the emmetropic eye corresponds to about the minimal focusing distance of a 50 mm macro 1:2 mounted in a “common” reflex camera. Then, with a camera, using (short) macro objectives we can focus near points shorter than emmetropic eye: myopic eye (myopic near point, see later) can be emulated, and also more.

[194] With also an excursion of the objective able to focus at 14.34 cm.

[195] $P_M > 1$ helps to increase the *d.o.f.*s. at the expense of a reduction of the *f.o.v.*.

[196] See Par. *2.1.2.1.2*: if the scene is static, with the collage method we copy the eye, that moving compensates for its poor foveal *d.o.f.*.

[197] See Par. *2.1.2.1.1.3*.

[198] Because paintings are normally realized by painters with emmetropic eyes (or ametropic but corrected eyes): myopic eye has a shorter *d.o.f.* than the emmetropic eye, that is compressed at near distances (the “focusing” distance is reduced to one too near point for normal paintings); hypermetropic eye has, again, a shorter *d.o.f.* than the emmetropic eye, but a far “focusing” distance that could be compatible to make a painting with the “Romantic effect”.

Optics causes the change of the depth of field, that depends also on the reproduction means (photosensitive element plus[199] screen or paper[200]) that influence the minimum acceptable amount of (absolute) out-of-focus. Optical contribution depends on M_{long} (so on the relative distance of the object (x_{object}/f)) (so *d.o.f.* extension is particularly short for macro objectives that can focus only a narrow longitudinal portion of the scene) and on *f#* (that, for definition, depends on the entrance pupil). For a wave optics treatment, see Par. *A.2.1.1*.

As highlighted in the *Introduction*, out-of-focus information could be used to derive the distances in a mono-objective photo[201]: it is necessary to know the amount of out-of-focus and f, P_M and E.

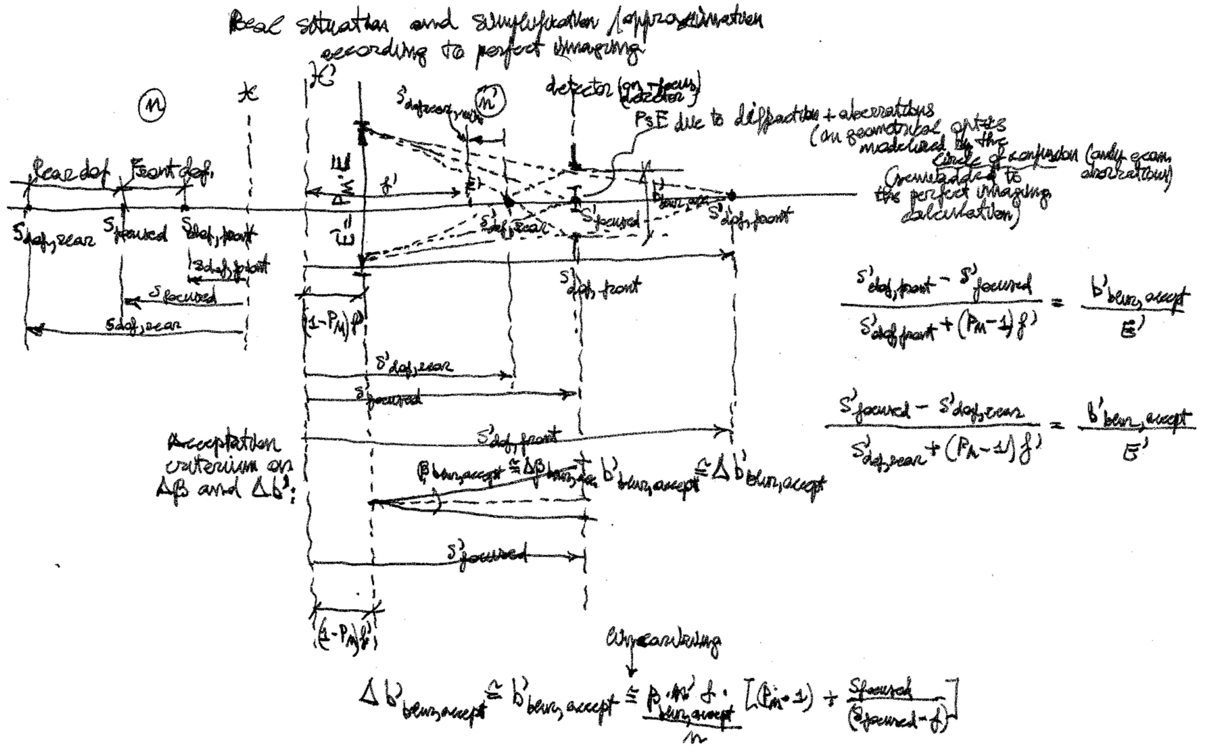
Real situation and simplification /approximation according to perfect imaging
n
n'
detector (on-focus detector)
PSE due to diffraction + aberrations (on geometrical optics modelled by the circle of confusion (only geom. aberrations) (reimbedded to the perfect imaging calculation)
rear dof
front dof
$s_{dof,rear}$
$s_{focused}$
$s_{dof,front}$
f'
$s'_{dof,rear}$
$s'_{focused}$
$s'_{dof,front}$
$b'_{blur,acc}$
$(1-P_M)f'$
Acceptation criterium on $\Delta\beta$ and $\Delta b'$:
$\beta_{blur,accept} \cong \Delta\beta$
$b'_{blur,accept} \cong \Delta b'_{blur,accept}$
$\frac{s'_{dof,front} - s'_{focused}}{s'_{dof,front} + (P_M - 1) f'} = \frac{b'_{blur,accept}}{E'}$
$\frac{s'_{focused} - s'_{dof,rear}}{s'_{dof,rear} + (P_M - 1) f'} = \frac{b'_{blur,accept}}{E'}$
Uncertainty
$\Delta b'_{blur,accept} \cong b'_{blur,accept} \cong \frac{\beta_{blur,accept} \cdot n' f}{n} \cdot \left[(P_M - 1) + \frac{s_{focused}}{(s_{focused} - f)}\right]$

Fig. A.2 - d.o.f. and acceptance criteria for a convergent system capable to give on-focus images (like the (on-focus) camera and the emmetropic eye), with $s_{focused} > s_{d.o.f., front} > f$ (for the ametropic eye and the camera out-of-focus, see Fig. A.3)

[...]

Fig. *A.2bis For the best point to focus on, to have the best d.o.f. according to the depth of the 3D scene: camera and eye cases.*

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[199] In the old cameras the photographic plate was directly seen, with an improvement due to the jump of one degrading passage.

[200] Its analog form or also its digital form, its "grain" (discretisation): pixel size or, at the limit, the quantum of information.

[201] Theoretically also the (single) eye could use the out-of-focus of the image on the retina, but the *d.o.f.* extension of the eye is very large, see later.

The ametropic (eye or camera[202]) objective (here simply a defocused system):

The ametropic eye/camera is generally an eye/camera with an optics affected by a refractive defect; here we will treat the out-of-focus of an axial object[203], that is due to an axial shift between $s'_{focused}$ and the the detector position, $\Delta s'$ [204]; if it is independent by the object distance, s , it is completely covered by myopia ($\Delta s'>0$) and hypermetropia ($\Delta s'<0$), otherwise, more generally, if a "defective" focusing[205] mechanism has an impact, it is named presbyopia (generally linked to short object distances[206], giving a $\Delta s'<0$).

This shift can be seen as due to three types of independent contributions; with regards in particular to the eye, we can speak about:

- Abnormal cornea curvatures[207] and/or aqueous humour refractive index and/or crystalline lens curvatures and/or vitreous humour refractive index[208], giving abnormal cardinal points values in the image side (for example, F' and/or H') of the four: (F, F', H, H')[209]. This contribution is said "refractive" contribution to myopia and hypermetropia.

- An abnormal position of the detector ("axial" defect): as seen in the picture below, the eye bulb length is longer/shorter than "normal" eye, refractive elements being the same, giving "axial myopia"/"axial hyperopia"). Usually this is the major (more spread) contribution to the myopia and hypermetropia.

- An abnormal focusing (accomodation) (named presbyopia), so "wrong" changes of curvatures (and consequently wrong f', H', etc.) vs. the object distance: normally with ageing these parameters give a $\Delta s'<0$ at short distances because the ciliary body muscle is not able any more to compress the crystalline lens. But is could be happens, for example, that, with exercises, with an "effort" it is more relaxed than necessary at medium distances, so that a myopic eye for the two upper reasons can improve, up to certain distances.

In the following, we will analyse in detail myopia only (so presbyopia is not considered), because hypermetropia can be derived similarly.

---

[202] We treat in this book the out-of-focus camera as if it would be an ametropic eye.

[203] For not-axially symmetric refractive defects that impact central zones, see, astigmatism (Par. *A.1.2.1.2*)

[204] Refer to Fig. *A.4*.

[205] We are speaking of the variation of the optical system curvatures/indexes-of-refraction, so about the eye that has a variable focal length focusing or the camera only if it adopts this strategy (with a zoom objective).

[206] The crystalline lens has difficulties to be curved, principally with ageing. But a $\Delta s'>0$ can be given by an excessive relaxation of the crystalline lens relative to a certain distance.

[207] The focus of axial point is in the paraxial geometrical realm, so we can treat only axially symmetric refractive defects that impact central zones of the eye/camera-objective optical surfaces, so curvatures.

[208] In the case of the camera

[209] See APP.2.

In the camera, the axial shift between $s'_{focused}$ and the detector position can be created artificially by a out-of-focus.

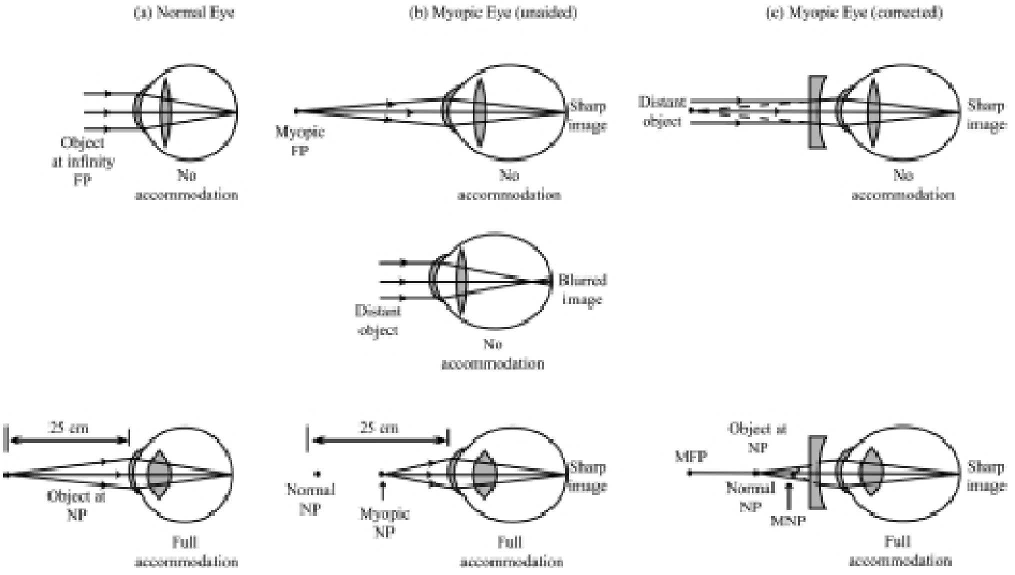


Figure A.3 – "Standard" emmetropic eye and myopic eye and hyperopic eye

Making reference to the Fig.*A.4*, we can analyse, in particular, the (de-)focusing of myopic eye (or the camera out-of-focus to create $\Delta s'>0$).

Myopic eye(/camera out-of-focus):

We have the detector displaced of $\Delta s'>0$ axially.

Some considerations:

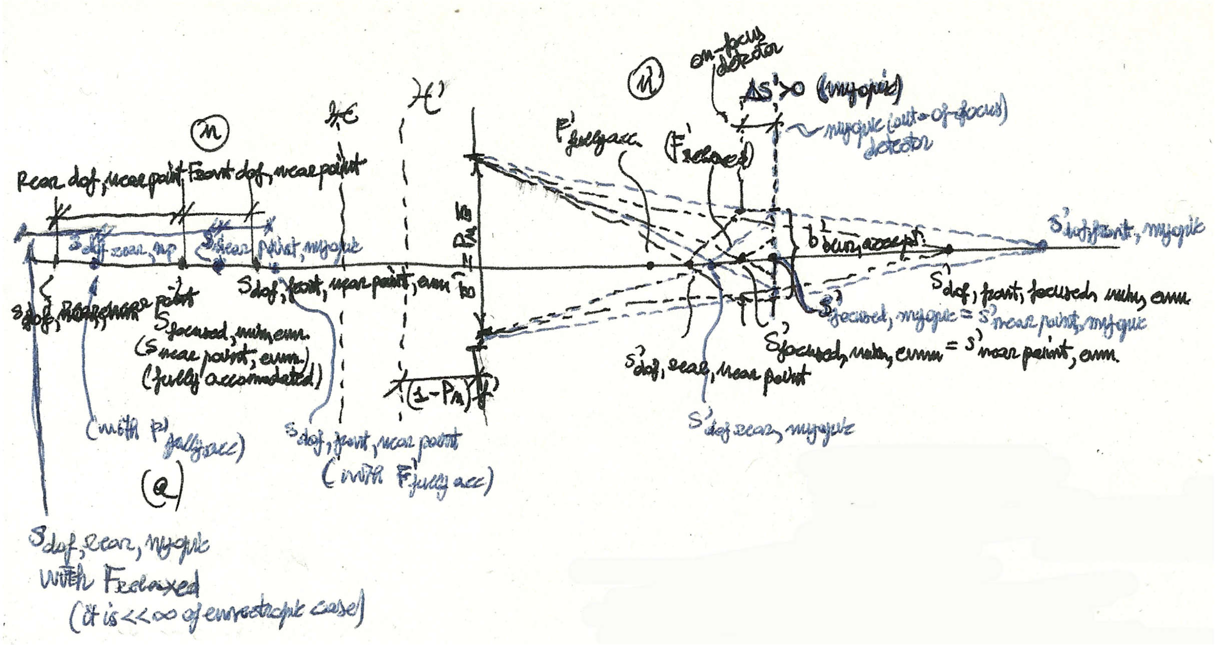
1. All objects are out-of-focus (see the blur that the $s'_{focused,\ min,\ emmetropic}$ forms on the "myopic detector"), except one single point, named "myopic near point" ($s_{near\ point,\ myopic}$), that is nearer than the $s_{focused,\ min,\ emmetropic} = s_{near\ point,\ emmetropic}$ (that is focused with the fully accomodated eye), so that is outside the emmetropic focusing range (but that can be farer that the $s_{d.o.f.,\ front,\ near\ point,\ emmetropic}$).

2. Then, the image of an object into the emmetropic focusing range will be always out-of-focus (see the cone in the picture). ….

3. *Front d.o.f.* and *Rear d.o.f.* for the myopic near point are both smaller than the ones for the emmetropic near point.

4. $s'_{d.o.f.,\ rear,\ myopic}$ brings to two different $s_{d.o.f.,\ rear,\ myopic}$, on with fully accomodated eye parameters, the other with fully relaxed parameters: this second one, $s_{d.o.f.,\ rear,\ myopic,}$

uncommodated is the maximum acceptable range of the myopic eye, and is << infinite of the emmetropic eye.



(n)
(a)
(n')
Rear dof, near point
Front dof, near point
$\mathcal{H}$
$\mathcal{H}'$
$s_{dof, rear, mp}$
$s_{near\ point, myopic}$
$s_{dof, front, near\ point, emm}$
$s_{dof, rear, near\ point}$
$s_{focused, min, emm.}$ ($s_{near\ point, emm.}$) (fully accomodated)
(with $F'_{fully acc}$)
$s_{dof, front, near\ point}$ (with $F'_{fully\ acc}$)
$s_{dof, rear, myopic}$ with $F_{relaxed}$ (it is << ∞ of emmetropic case)
b_0
$\Delta s'$
$(1-P_n)$
$F'_{fully acc.}$
($F'_{relaxed}$)
on-focus detector
$\Delta s' > 0$ (myopic)
myopic (out-of-focus) detector
$b'_{blur, accept.}$
$s'_{dof, front, myopic}$
$s'_{dof, front, focused, min, emm}$
$s'_{focused, myopic} = s'_{near\ point, myopic}$
$s'_{focused, min, emm} = s'_{near\ point, emm.}$
$s'_{dof, rear, near\ point}$
$s'_{dof\ rear, myopic}$

Fig. A.4 – d.o.f., myopic near point, hyperopic far point for a convergent out-of-focus system (like the ametropic eye and the out-of-fpcus camera), with $s_{d.o.f., front} > f$ (for the emmetropic eye and the on-focus camera, see Fig. A.2). (a) for the myopic eye (in ***blue****), in particular the case of fully - accomodated eye is detailed; (b) for the hyperopic eye (in* ***green*** *the hyperopic case).*

Numerical examples for the myopic eye and for the out-of-focus camera:

Axial myopia vs. emmetropic eye:

……

Refractive myopia vs. emmetropia:

……………

A specific myopic eye and the camera with the same out-of-focus degree:

Supposing no astigmatism, my left eye has …. Dioptres.

Supposing that my ophthalmic lens is positioned at the Knapp distance and there corrects perfectly my myopia (so I can see 10/10 at the Snellen chart) and that I have a “normal” accomodation mechanism, $\Delta s'$ = … mm.

With two cameras, one having a fixed focus, the other having a zoom, so the same focusing mechanism of the eye, and with the zoom having the same principal points (H' and F') of the eye and that moves around of the same range. …..

A.1.2.1.1.2 ***Dynamical and statical focusing***

Camera autofocus vs. eye autofocus: the two autofocus are based on a closed control loop, but the eye makes also little “lateral” movements to improve autofocus.

Camera bases focusing on longitudinal objective movement with a fixed focal length, whereas the eye changes crystalline curvatures, so mainly focal length (and rear unitary plane position) (see Appendix *APP.2.1.1*).

(Video-)cameras have a temporal performance that can be better than eye, but in dynamic autofocus video-cameras they are slower than eye+brain.

A.1.2.1.2 The five (purely spatial) aberrations

The first three aberrations (spherical, coma and astigmatism ones) generates, in the ideal realm of geometrical optics, not an image point but a spot, so the projective transformation of perfect imaging is not valid any more to describe the transformation by optics.

- **Spherical aberration**
- **Coma**
- **Astigmatism** To reproduce reality, so to (forcedly) image a point in a “spread” point, the focal length (so the dioptric power) must be the same along each meridional plane[210] of the optical system (axial revolution symmetry achieves it, but it is not demanded), then the optical system has to be non astigmatic. So, in an ideal objective, the optical power is isotropic, i.e. equal for each meridional plane, and it will be indicated by the scalar P [Dioptres]; but in any real objective the optical power is described by a tensor[211] because it is affected by astigmatism (for example, for a single lens as the cornea, corneal topography can well depict the non-constant dioptric power[212] of the first surface of the cornea): to correct for astigmatism, one simple ophthalmic cylindrical lens is used, forming, if a spherical disease as myopia or hyperopia is present, a toric lens; but it can not, in general, correct completely for the astigmatism: the eigenvalues of the 2x2 matrix (magnifications or powers) are not in general orthogonal each one (the toric lens completely corrects the astigmatism only if the eye meridional planes of max. and min. power are fortunately orthogonal[213]).

---

[210] More precisely, the mean value along the meridional plane must be the same

[211] Supposing that the optical system is not an axial revolution one, but conserve an axis, the image is always magnified/mignified longitudinally (along the axis) in a constant manner (according to M_{long}), but transversely by a not diagonal 2x2 matrix for magnifications and for powers (focal lengths).

[212] Due to both these factors: no-rotational symmetry of the curvature of the two corneal surfaces and no-rotational symmetry of $n_{cornea}(\lambda)$. Normally, only no-rotational symmetry of the first curvature is measured by the corneal topography.

[213] So if the matrix is already diagonal, that is the usual case.

- **Curvature of field** .....
- **Distortion** ….

Camera objective

….

Eye objective

….

A.1.2.1.3 Chromatic aberration and colour reproduction

World is coloured, not monochromatic, so practically chromatic aberration has always to be taken into account. As said, it is due to the influence of system temporal response on its spatial one (through the phase speed): the temporal spectrum widening in time is transferred in a space spreading when the point is composed by a not punctiform temporal spectrum[214].

Colour reproduction, instead, is the direct influence of the system temporal response in time domain (time point spread function that brings to a widening of the original time spectrum). It is due to the time frequency response of the optics and to the punctual colour response of the detector.

A.1.2.1.4 **Sharpness evaluation using wave optics**

A.1.2.1.4.1 The MTF of the eye and of the camera optical systems

Herein the $MTF_{eye,optics}$[215] is graphed: due to aberrations, the best $MTF_{eye,optics}$ is obtained with an entrance pupil diameter of 2.4 mm. Instead, camera objectives, above all tele-photo objectives, thanks to multiple lenses, can improve diffraction limit when the pupil diameter is increased a lot, so they arrive to a much better *MTF* than the eye one[216]: compare the following picture.

---

[214] A pure frequency line spectrum is an ideal case.

[215] Differently from $MTF_{camera,optics}$, $MTF_{eye,optics}$ is hard to measure because it must be extracted by the $MTF_{eye,total}$ that comprises also the $MTF_{eye,retina+brain}$: the quantity that is really directly measurable by tests on persons is is the contrast *sensitivity* at each frequency for a certain retinal illumination and is influenced by optics+retina+brain,

[216] We could put a camera objective in front of the retina, but it would be cumbersome because it should be bigger than the eye bulb.

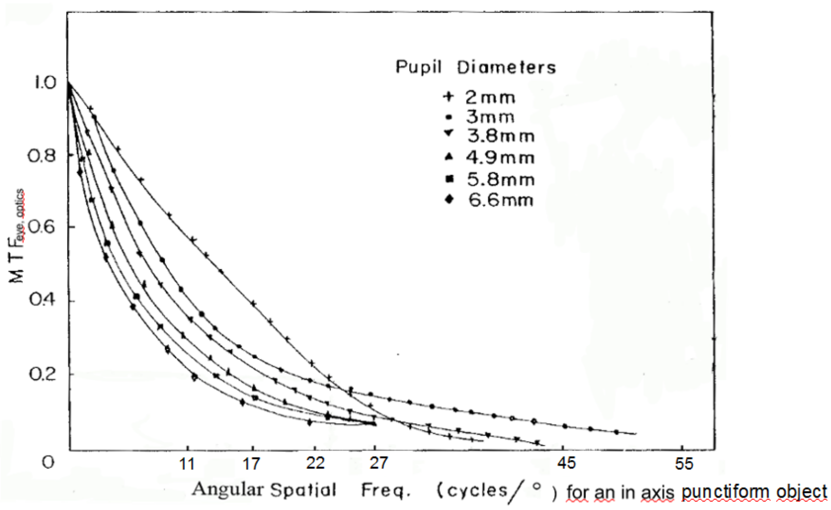


Fig. A.5 - MTF$_{eye,optics}$ with white light (from a high-pressure mercury lamp) for an in-focus object at various entrance pupil diameters. For the eye, there is not a true distinction between an in-axis punctual object and an object that covers all the narrow field of view (for off axis object points there is coma, and very few vignetting).

MTF for optics alone of a good tele-photo objective, in [cycles/°]

Fig. A.6 - MTF$_{camera,optics}$ with white light (from a high-pressure mercury lamp) for a good tele-photo objective in focus, for various f#. For the camera, there is a distinction between an in-axis punctual object and an off-axis point object, because the f.o.v is high: an MTF$_{camera,optics}$ graph would be necessary for each object-point angular direction (for off axis object points there is coma and can be present some vignetting).

Fig. A.7 – Fresnel P.S.F.. for a refractive defective eye (myopic or hyperopic) is the same because Fresnel P.S.F. depends on the module of dioptres)

D.o.f. treated with wave optics makes use of the $|\mathsf{F}^{-1}[\text{Fresnel } MTF_{optics}]|$ = *Fresnel Line Spread Function.*

A.1.2.2 Energetic evaluation using geometrical optics

Energetic evaluation (in particular entrance and exit pupils) can not be treated by paraxial optics (in which there is not entering power because entrance pupil has zero extension).

A.1.2.2.1 The entrance (/exit) pupil (geometrical optics is enough)

The light amount that enters into a *camera obscura* or into a *pinhole camera* depends on the aperture stop, that, in this case, is unique[217] and well determined. Otherwise, the light quantity that enters into an optical system is related to the entrance pupil[218] that, due to the optics, doesn't coincide (in position and dimensions) with the exit pupil[219]. The exit pupil is useful, for example, if another optical system in cascade must be connected (eye comprised), to match the entrance pupil of the following system to collect all exiting energy[220]; or it is useful to refine[221] *f.o.v.*, *d.o.f.* and effective aperture calculations.

Stated the optical system, pupil's diameters and positions (four degrees of freedom) can not be all defined, because we have only two[222] degrees of freedom (the diameter and the position of the aperture stop). The ratio of the dimensions of the pupils, said pupil magnification, P_M, can be one degree of freedom, and is determined by the aperture stop position; then the absolute diameter of one of the pupils can be decided.

---

[217] The aperture stop coincides with E and E', see Fig.16bis.

[218] For off-axis object points, a part of the entrance pupil, and/or a part of the exit pupil can be "obscured" by vignetting, that decreases the light amount gradually starting from a certain angle (see Par. *A.1.2.3.3*).

[219] The entrance/exit pupil dimensions are forcely limited, at least, by the aperture stop constituted by one of the optical surfaces; but there is the possibility to add two additional degrees of freedom: a variable aperture stop can be placed between the first-last surfaces to reduce the entrance/exit pupil dimensions and to put them in a relative position.

[220] The exit pupil can be littler or greater than the entrance pupil (P_M = *Exit Pupil* / *Entrance Pupil* = E' / E is the pupil magnification (always positive, see Note 222)). Normally, in systems that must be used directly by the eye, it is of the same size of the (entrance) pupil eye (for example, in telescopic systems exit pupil is littler than entrance pupil, being the entrance pupil large to improve diffraction limit, so the input light is concentrated into the smaller exit pupil).

[221] See the dedicated paragraphs.

[222] We can choose, for example, diameter of the entrance pupil (important for the light gathering) and position of the exit pupil (for *f.o.v.*, *d.o.f.* and effective aperture).

The special case of $P_M = 1$ means that the aperture stop has been placed in a "symmetrical way"[223] because the (transversal) magnification of the aperture stop by the left part of the optical system is equal to the magnification of the right part: in this case also the distances of the pupils respect to the aperture stop are the same (if $n = n'$).

The (variable) eye entrance pupil (in common speech named "pupil") is slightly bigger than the aperture stop (diaphragm) of the eye (the iris)[224], that is positioned between the two eye's lenses, whereas the photographic objective pupils are determined by a (variable) aperture stop[225] that can be putted in various positions, normally between the first and last surface.

$P_{M,Eye,\ uncommodated} = 0.92$[226]

$P_{M,camera}$: in fixed focal length objectives, it doesn't depend on object distance, but it depends on the objective focal length and on the constructive type of objective.

Differences in performance due to the position of diaphragm are......

The *f-number* of the eye varies from TOT to TOT, the *f-number* of a photographic objective can varies from 1 (for certain high speed fixed 50mm objectives) to 22. The range from 1 to 22 is much higher than eye (=…..) because cameras normally don't have dynamic range as great as the eye (retina compensates for the lower *f-number* range). But camera sensor can be setted to different sensitivities.

You can see the effects (for example if you see yourself in a mirror) without to arrive to the pupil.

…………………………

A.1.2.2.2 The field of view and its origin ($P.o.V.|_{3D\ scene}$) (geometrical optics is enough)

The origin of the *f.o.v.* (the $P.o.V.|_{3D\ scene}$ of the eye, see also perspective in Appendix) is the centre[227] of the entrance pupil[228] of the optical system. Its (angular) amplitude (*f.o.v.*) is delimited by the entrance window[229], so it is determined by the field stop[230], by optics or by supplementary mechanics, or by the detector dimensions (in the camera the (fixed) field stop is the detector, in the eye, for the total visual field of view, it can be the eyelids (→ variable field stop (→ variable *W* (and variable *W'*))) or, if the eyelids are open, orbits plus nose (→ field stop and windows fixed); if we consider only the central *f.o.v.* (visual field of view), by the fovea dimensions (detector)). The *f.o.v.* is the maximum possible when the field stop is the detector itself; in this case, its value varies with object distance and, by convention, its value is its maximum value, so when object is at ∞ (see Fig. *A.8*).

---

[223] Only respect to (transversal) magnification: it doesn't mean automatically that powers at its left and at its right are the same (symmetrical), but only that transversal magnifications are the same (symmetrical).

[224] See Appendix for calculations.

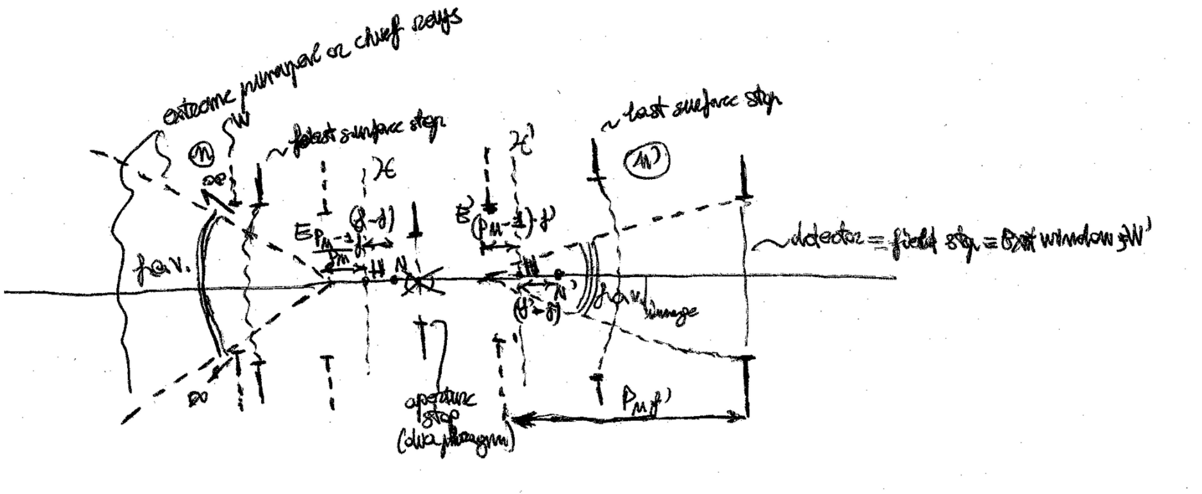
[225] Also named diaphragm.

[226] It depends on accomodation. See Appendix, page 87, for calculations.

[227] The entrance and/or exit pupil extension can cause vignetting (see *A.1.2.3.3*).

[228] There are two cases in which it coincides with the first nodal point: in paraxial optics model (and when *f.o.v.* is little: N can approximate *P.o.V.*, see Appendix.) and when pupils are positioned at the nodal points (when *Pupil Magnification* = 1 AND $n = n'$). In the case of the eye, for foveal vision, because *f.o.v.* is very small and P_M is around 1, *P.o.V.* is not very far from N.

Eye can also move, so total field of vision is wider, and the photo should cover entirely this wider extension (with ratio 16:9, as the ideal eyeglass lens and as cinema[231], see the figure).



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229 We put the "observer" in the object hemispace, so *E* and *W* are used. To use the aperture stop we would define a $W_{field\ stop}$ seen by the aperture stop.

230 That fixes the entrance window, *W*, and the exit window, *W'*.

231 Panoramic cinemas try to cover this extension.

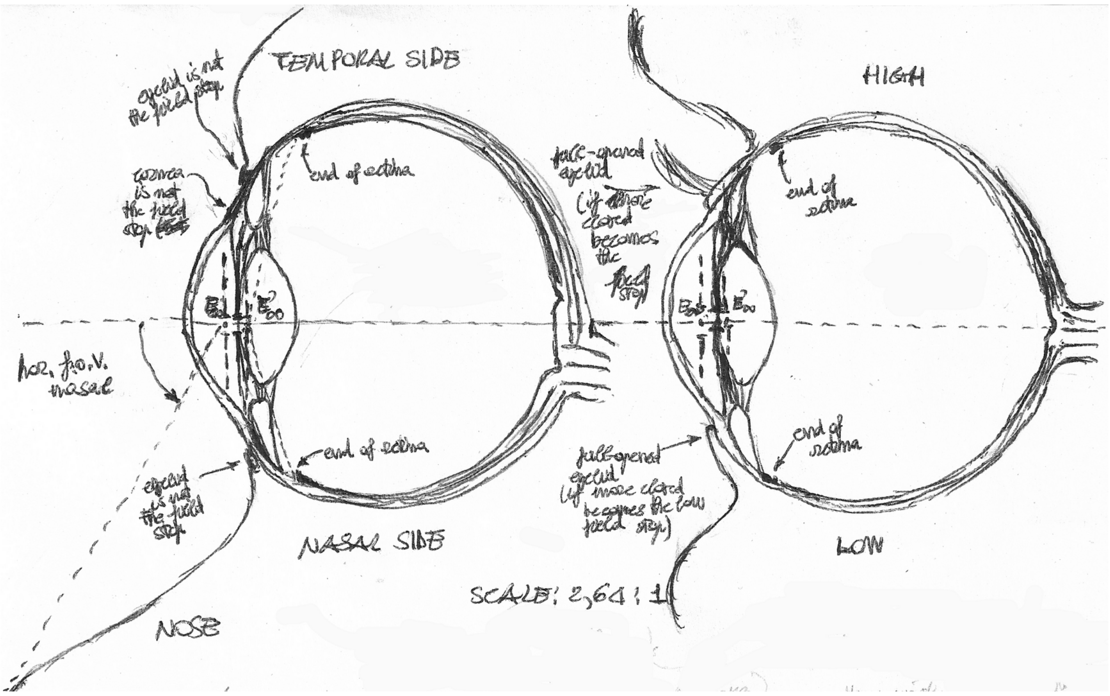

TEMPORAL SIDE
eyelid is not the field stop
cornea is not the field stop
end of retina
E_0
E'_∞
hor. f.o.v. nasal
end of retina
eyelid is not the field stop
NASAL SIDE
NOSE
full-opened eyelid (if more closed becomes the field stop)
full-opened eyelid (if more closed becomes the low field stop)
SCALE: 2,64 : 1
HIGH
E_0
E'_∞
end of retina
end of retina
LOW

Fig. A.8 – The upper picture (with fixed W' corresponding to the detector) can be used for the camera putting n = n'; the second is for the horizontal and vertical (uncommodated) eye f.o.v. with centred eyeball.

As in …. Appendix, with $M_{transv} = P_M$ (see also the figure above), *f.o.v.* and $f.o.v.|_{image}$ are not, in general, the same[232]:

$$\tan(f.o.v.|_{image}/2)/\tan(f.o.v./2) = f\,/\,(f'\,P_M)$$

For the uncommodated eye:

$$f.o.v_{Eye\,,uncommodated} = 2\tan^{-1}\left[\frac{\left(\tan\left(f.o.v._{Eye,\,image,\,uncommodated}/2\right)\right)f'_{Eye\,,uncommodated}\,P_M}{f_{Eye\,,uncommodated}}\right] =$$

$$2\tan^{-1}[1.2291\,\tan(f.o.v._{Eye,image,uncommodated}/2)]$$

So $f.o.v._{Eye,\ uncommodated}$ is slightly smaller than $f.o.v._{Eye,\ image,\ uncommodated}$.

During taking a photo the centre of E_{camera}[233] can coincides with the centre of $E_{eye} = P.o.V.|_{3D\ scene}$ only if $P_{M,\ camera} = 1$, otherwise a correction must be made (see formula above) to allow, during the vision of the photo, to place the centre of E_{eye} coincident with the centre of E'_{camera} to respect perspective/projection[234].

---

[232] So for systems with *f.o.v.* limited by *W'* (as camera, with *W'* constituted by the detector), the *f.o.v.* is only determined by detector dimensions and by *f'*, and not also by P_M, also if P_M is different from 1: the change of $f.o.v.|_{image}$ due to $(1-P_M)f'$ is compensated by the $f.o.v./f.o.v.|_{image}$ magnification. The general formula for the *f.o.v.* extention is: $\tan(f.o.v./2) = (d/2)/f$. For example, a 35 mm objective mounted on a full-frame (24mm *x* 36 mm) detector camera will have a *f.o.v.* (vertical or horizontal) always equal to: $2\tan^{-1}$[(12mm or 18 mm) / 35 mm] = 37.8° *x* 54.4°.

[233] With or without the trick of the "collage".

Important note (uniform and non-uniform field of view (intensity field of view)):

Not all the *f.o.v.* is illuminated as the object source, due to optics (aperture efficiency, vignetting, cos^4, etc.) and sensor (not uniform sensitivity) contributions. In the eye the main contribution is due to the retina sensitivity inhomogeneity, but the eye instinctively moves with saccade movements to shift the object of interest in the foveal region. See later for a discussion about energetic aspects.

A.1.3.3 Attenuation

….

A.1.3.4 Aperture efficiency

….

Aperture efficiency is 1 if the lens is a thin lens

Camera objective:

....

Eye objective:

….

A.1.3.5 Vignetting

Definition and considerations:

(Optical) vignetting decreases the energy of lateral object points that are inside the *f.o.v.*, so in this case the *f.o.v.*, from an energetic point of view, is “full” only into a smaller angular range.

---

[234] But see previously for the necessity of the trick of multiple photos with a focusing based on the zoom of the camera to really respect natural perspective.

It is present if the entrance and/or exit windows are coincident to one of the surfaces of the optical system[235]. If not, marginal rays of an angularly lateral object (near the limits of the *f.o.v.*) can be not transmitted by the lenses[236] when the pupils have a “too” high extent respect to their diameter[237], causing a luminosity decrease at the edges of the image. This drawback can be acceptable, so used, to reduce off-axis aberrations at the edges of the photos in high *f.o.v.* objectives, for example in wide-angles at high *f#*[238].

Camera objective:

....

Eye objective:

....

Fig. A.9 – Vignetting in camera obscura and eye (non foveal visual field)

A.1.3.6 Relative illumination

---

[235] So it is present in the eye, where the eyelids are the field stop (and the entrance window, *W*). In the eye vignetting can change and can be not symmetric because *W* changes, according to the eyelids opening and to the bulbe movements: you can test it by seeing the decrease in luminousity of lateral objects by widening the eyelids (also: ophthalmologist says you to open your eye when he sounds out the side parts of the ocular fundus) (see also its possible influences on visual field measures); consider that the result with only moving the eye are magnified by the photoreceptor density.

[236] This is due to the not null extent of the pupils, see A.1.2.1.1, when the lenses constitute, globally, a convergent sub-system.

[237] If the field stop is the detector (the case of the camera), the lenses *after* the aperture stop are the involved ones, otherwise if the field stop is in object hemi-space, the lenses *before* the aperture stop are involved.

[238] So the diameter of a camera objective, that is proportional to the (entrance) pupil (so to the focal length and to the *f#*) and to *f.o.v.*, can be less than previewed due to the *wanted* vignetting at high apertures.

Definition and considerations:

Influenced by aperture efficiency, vignetting and cos^4. It depends also on the sensitive uniformity of the sensitive layer.

Also the diffraction has an influence, because there is a hole that diffracts, and not free space.

Camera objective:

....

Eye objective:

….

A.1.3.4 Energetic evaluation using wave optics

Parameters not included in *MTF*: Attenuation, Aperture Efficiency, Vignetting, Relative Illumination, (Flare partially) because the *MTF* is calculated for a central point source; besides, if the *MTF* is not calculated for the real spectrum of the source, Chromatic Aberrations and Colour Reproduction .

In any case, the analysis by wave optics gives very little changes to the energetic evaluation.

A.1.3.5 Colour reproduction

Word is in colours. The three parameters of colour reproduction are:

- Hue
- Luminosity
- Saturation

Optics can absorb differently the colours, or shift them. In the following also the sensor contribution to colours will be considered.

A.1.3.6 Flare

Definition and considerations:

Flare is due to reflections at lens interfaces. The result is a decrease of the modulation[239] due to the background light or, in old objectives, multiple luminous rounds in the image. This phenomenon is increased if a high luminous source enters in the *f.o.v.*, for example sun light. To try to reduce it, a lens shade must be used (in the case of the eye, make this experiment: try to see a grass with sunshine and, after using your hand as a “lens shade”, you will see a more sensational scene (contrast[240] will be higher)).

Camera objective:

....

Eye objective:

The eye has limited flare because of the presence of only two lenses (four interfaces). And because of the low difference (lens-medium) between index of refraction of lens-surface/medium[241].

But it has not a lens shade (eyelids are not enough).

A.2 The detection systems of the eye and of the camera

For a complete performance analysis also the visualisation system of the camera luminous output (paper, monitor, etc.) must be added[242]. On the other hand, to continue on the comparison, the detector+visualisation system alone must be compared to the retina (+brain = optical nerve and visual cortex).

A.2.1 The sensitive layer (+shown means) of the camera and the retina (+optical nerve and visual cortex)

A.2.1.1 Their contribution to the frequency response

Spatial performance in terms of $MTF_{camera,sensor+shown\ means}$ and $MTF_{eye,retina+\mathrm{brain}}$ can be only an approximation in a limited range of source intensities, because camera sensor + shown means and retina+brain are not linear as, instead, at a high extent, optics (of which absolute and relative sensitivity and dynamics are not specified): $MTF_{camera,sensor+shown\ means}$ and $MTF_{eye,retina+\mathrm{brain}}$ are not a unique characteristic function for the system, but they should be many functions[243], each one calculated for the different light intensities of the source.

---

[239] From the information perspective, a decrease of the Signal to Noise Ratio, as the one due to the noise.

[240] Your hand creates a field stop but only on lateral *f.o.v.* and not on central (foveal) *f.o.v.*.

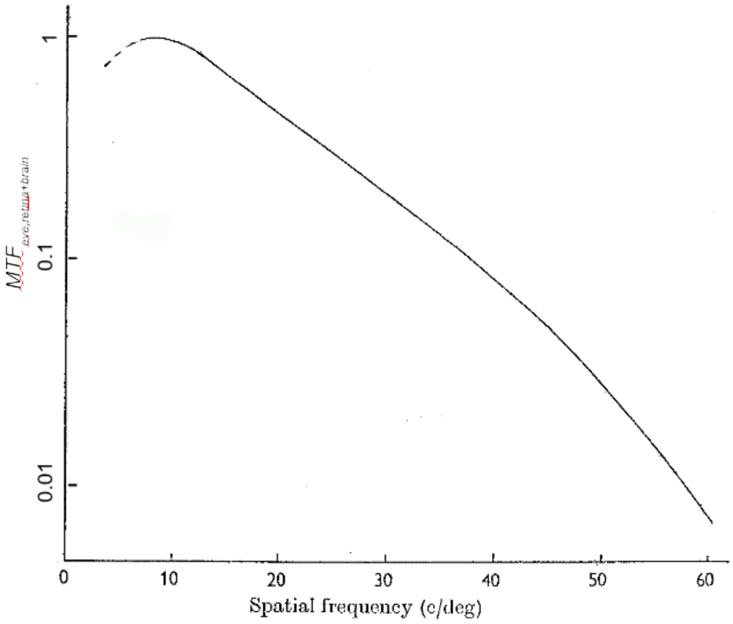
[241] Fresnel reflection depends on this difference.

[242] Because the sensor can not be seen directly, as in slides and old photographic plates.

[243] Theoretically for all the infinite intensities of the source, practically for all the discretised values of the source intensity.

Whereas the $MTF_{eye,optics}$ is worse than $MTF_{camera,optics}$, the $MTF_{eye,retina+brain}$ is better than present $MTF_{camera,sensor+shown\ means}$. $MTF_{eye,retina+brain}$ can be derived[244] by the size of one of its optical contributor, the "pixel" of the retina (the cone), equal to 3 μm, that is less than the pixel size of the present camera sensors, that are less dense: nevertheless, to compare them it is necessary to pass from the linear resolution to the angular resolution that depends on the position of the detector respect to the exit pupil, so a cone subtends, in the uncommodated eye, …., whereas the pixel size of present cameras subtends a variable angle (for far objects) that depends on f (…. for a 20 mm focal length, … for a 100 mm objective) (see the following pictures for a final comparison). The "contribution of the pixel size" ($MTF_{eye,retina+brain}$ or $MTF_{camera,\ detector+showing\ means}$) to the total MTF ($MTF_{eye,total}$ or $MTF_{camera,total}$) depends[245] on $f\#_{Eye}$ or $f\#_{camera}$.

Total $MTF_{eye,total}$ is worse than the $MTF_{camera,total}$ due to the very good $MTF_{camera,optics}$. In any case, the eye was made to see directly the "reality", not a photo or itself[246].



---

[244] According to the Nyquist theorem.

[245] If $f\#$ is lower, the pixel size must be smaller. For a good MTF_{total} a high E must be used, but the pixel must be reduced in function of $f\#$ (for high speed objectives the pixel size must be reduced). We assumed the most unfavorable case of diffraction limited system; if not, the constribution of the pixel size is less influent. Taking fixed f and $f\#$, so the $MTF_{camera,\ total}$, to increase the information quantity, and also to increase the *f.o.v.* of the single shot, an increase of the detector size (so the passage to another, bigger, format) is necessary.

[246] Also during dreams we see memorised images, so directly the reality as was seen by the eye (there is not a double pass through the eye).

Fig. A.10 - MTF$_{eye,retina+brain}$ with monochromatic light at 530 nm with the best condition of high retinal illumination (500 Trolands). This MTF is valid only with this intensity and is approximated when applied to a real scene with its intensity and frequency spectra (see in the text)

Fig. A.11 - MTF$_{camera,\ detector+showing\ means}$. Retina probably is better. Long focal lengths[247] *of the objective in non-telescopic systems, and detector far from the objective in telescopic systems, improve it ["granularity" of the detector gives less bad contributions*[248]*], so the function MTF$_{camera,\ detector+showing\ means}$ depends on f .*

[...]

Fig. A.12 -Pixel density for camera detector and for retina

---

[247] In the eye the better resolution is for maximum *f'*, so for uncommodated eye (Rayleigh criterion is applied to the nuke eye for distant objects, "the stars"). The same is true for showing means: it is better to see the photo from far to decrease the"granularity" of the showing means.

[248] With short focal lengths objectives, with high luminousity and good MTF little pixels can avoid to be the limit of the MTF and can avoid to worse too much the global sharpness and information quantity (that can go to be limited by optics), instead for long focal objectives (that have a high entrance pupil and in general low aberrations (→ good MTF)) the effort to have little pixels do not give great advantages (the limit to information quantity and to sharpness is surely the optics).

The calculation of the total *d.o.f.* using wave optics makes use also of the $|F^{-1}$[Fresnel $MTF_{camera,detector}$]| or $|F^{-1}$[Fresnel $MTF_{eye,retina+brain}$]|

A.2.1.2 Their contribution to intensity fidelity (dynamical range/linearity)

....

A.2.1.2.1 Sensitivity, linearity and dynamics (and noise)

Optics, with its power losses, can contribute to the sensitivity, but very low to the luminance response curve (dynamics), because optics has a high linearity and a high dynamical range. On the contrary, sensor has an impact on power response curve.

Eye detector (retina+brain):

The eye detector has a very good sensitivity in dark, also if the spatial [and temporal] resolution decreases: they are correlated when we go towards to the limits (sensitivity and spatial resolution), because in this case the limitation of all of them is the noise[249]. The retina+brain has a good linearity curve and also high dynamics, that is, a good total sensitivity range.

Camera detector + reproduction means:

The camera can have a higher sensitivity[250], also if the spatial and temporal[251] resolution decrease with darkness, due to the noise. In any case, linearity curve and dynamics can be degraded vs. eye when sensitivity is very good. The sensitivity permitted by a photo can be better than the eye (the camera detector amplifies a very weak signal), but, because the eye is the final mean to see the photo, linearity, (relative) sensitivity and dynamics can not be increased by a photo. The same for spatial *x* temporal *x* intensity resolutions (they are expressed by the global *MTF* and, finally, limited by the noise).

---

[249] We are neglecting the quantum limitation, in the supposition that we stay far from this limit.

[250] When a photo is taken in darkness, with a very high exposure time, we can catch particulars that are not visible to the eye.

[251] See the increase of ISO sensitivity of the camera detector and its increasing noise amount. The rods of the eye are greater than cones.

A.3 The eye and the camera as a complete system

A.3.1 The performance of the human vision and of the photography

…..

A.3.1.1 Total performance of the eye

….

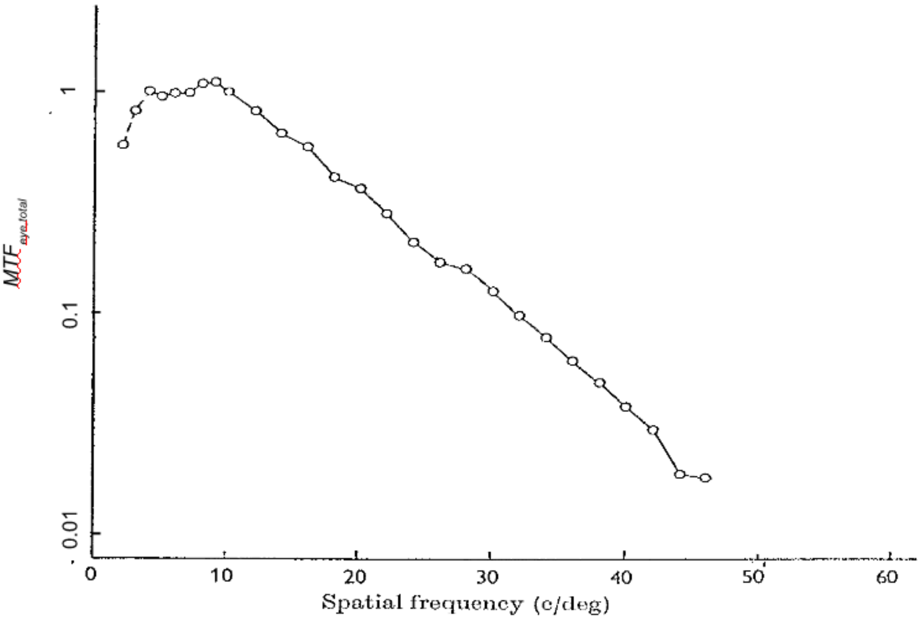


Fig. A.13 - MTF$_{eye,total}$ *with monochromatic light at 530 nm, pupil diameter of 2 mm and with high retinal illumination (500 Trolands). The MTF is valid only with this intensity and is approximated when applied to a real scene with its intensity and frequency spectra (see in the text). It is the composition of the optics and retina+brain MTFs. Adapted from Campbell and Green, 1965.*

A.3.1.2 Total performance of the camera+showing means

Fig. A.14 - MTF$_{camera,total}$ *excluding the showing means*

Fig. A.15 - MTF$_{camera,total}$ *including the showing means*

PART B: CAMERA+EYE vs. EYE[252]

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[252] See one of the piece of art by Andrea Paolini that belongs to the cycle on the vision: https://lartedelsorriso.eu/portfolio/ciclo-sulla-visione/ that is possible only with reflex cameras, not with mirrorless ones.

B.0 Introduction

We will treat camera+eye vs. eye when the eye sees directly the scene, so when the photo is compared to the reality. $MTF_{camera,total} = MTF_{camera+shown\ screen}$ doesn't exist alone, but only as an input to the eye+optical nerve+ visual cortex, so as a contribution to the global $MTF_{camera+eye}$.

B.1 Camera+eye vs. eye when taking a photograph

Do not have embarrassment in movements and in real life, the camera would be very compact, ideally of zero dimensions. Compact cameras exist, especially mirrorless ones, video cameras also; cinema and TV cameras are instead bigger, transforming the video in a non-natural way of seeing. Also very compact cameras exist, of dimensions similar to the eye, but they have not the required good characteristics, so they are not much better than the eye.

For example, the not coincidence between the centre of the entrance pupil of the eye and the centre of the entrance pupil of the camera objective determines that the photo doesn't restitute exactly the same (natural) perspective of the photographer[253].

Fig. B.1 - Reflex, mirrorless and a new architecture camera and the coincidence between its $P.o.V.|_{3D\ scene}$ *and the eye* $P.o.V.|_{3D\ scene}$ *when taking a photo*

B.2 Total performance of the artificial eye ((camera) objective+retina+brain)

Not considered here.

---

[253] Not always we want to see as the photographer, but only as if we would be there, because some *P.o.V.* of the camera are impossible for a human.

B.3 Total performance of: total camera + total eye vs. eye[254]

…

B.3.1 More degraded characteristics and the ones less degraded by a photo

Some of the characteristics to evaluate (to judge) a photo are more degraded after the passage through the eye, others are lower. For geometrical characteristics judgement, we started using the *MTF* of the complete camera system (optics+detector) added to the visualisation system contribution (paper or screen, etc.). After, considering par. A.3.1 (Total performance of the eye) and par. A.3.2 (Total performance of the camera), we can evaluate which parameters cited in par. A.3.1 are more worsened:

- *f.o.v.*: normally *f.o.v.* is strongly reduced by photography (it is less than the total field of vision: 180° x ...120°)

- Perspective (depth's reconstruction): it is not degraded if *P.o.V.*s ($P.o.V.|_{3D\ scene}$ and $P.o.V.|_{photo}$) coincide, then if the centre of the entrance pupil of the spectator eye (E_{Eye}) coincides with the entrance pupil of the camera objective (E_{camera}), but they generally don't exactly correspond: problems appear with wrong magnification of the photo vs. distance from which the photo is observed, if the scene has a high depth and if eye moves when seeing the photo (because in both the cases E_{Eye} moves), etc.

- Real distances: if the foreground object (nearer object) has 1:1 scale, its[255] distance is as the reality.

- The depth of field (Par. A.1.2.1.2.1): because the photo is planar, practically there is no degradation added by the eye [except for the corners that are at a greater distance vs. the *PoV* projected on the photo plane]; but if the photo is taken by only one shot (without the trick of the puzzle), the *d.o.f.* can be degraded a lot vs. reality seen directly by the eye.

- Geometrical image quality: the five aberrations (Par. A.1.2.1.3) should be null in the reproduction (photo or screen) → impossible; taking into consideration also diffraction: $MTF_{camera+eye} = MTF_{camera}\ x\ MTF_{eye} \rightarrow MTF_{camera}$ should be = 1 for all frequencies → impossible. So geometrical image quality is degraded, in a small manner with good cameras.

- Parameters not included in the final $MTF_{camera+eye}$[256]: absolute illumination, noise, absolute and relative sensitivity and chromatic sensitivity, dynamics, etc.: of them, absolute illumination and sensitivity can be increased seeing a photo!

---

[254] It would be interesting to treat the case of the camera connected to the optical nerve or to the visual cortex [see researches by W. H. Dobelle]

[255] This only.

[256] Gain-(Temporal)Bandwidth and Gain-(Spatial)Bandwidth, so the area of the *absolute MTF*, is of interest above all when passive negative feedback is applied.

B.3.1.1 Resolution (Space-Time[257])

Staying only in the spatial domain, final *d.o.f.* of the photography depends also on the grain (quantum of the reproduction means: photosensitive element and screen or paper), see Par. *A.2.1.1*.

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[257] In particular, Space *x* Time.

PART C: ACHIEVING THE IMPRINTING THE REALITY

C.0 The means of photography

Photography is the achievement the registration/reproduction of the spatial visual content (wavefronts and/or distributions of photons) of reality. For doing it, it is necessary to use a recordable optical photosensitive mean and a showing mean. They can coincide, for example, in a film (conversion of optical waves into chemical variations) (after, the photographic paper is only a showing mean), or to be separated, as in optoelectronic devices, like CCD or CMOS (conversion of optical waves into electromagnetic waves of lower frequencies), later connected directly to the visual cortex or coded on a optical or magnetic support and finally showed to the eye.

C.1 Philosophy

We need to compare camera+eye to eye alone. Unfortunately, during taking a photo, camera influences the viewing of reality (see Par. B.1) and, above all, also if perspective rule (*P.o.V.*s) was respected, it creates a copy of reality that can not be equal to reality: $MTF_{eye,total}$ can not be improved by seeing a photo.

If *P.o.V.*s position rule is *not* respected (the image “depths” are *not* correct) and if we are restricted to a particular object[258], (emmetropic) eye resolving power (that is related to $MTF_{eye,total}$) limitation can be overpassed[259], by the *angular magnification* of the image, so:

1. Taking a photography of the scene and then approaching, in seeing it, more than the perspective constraint, or, that’s the same, by magnifying too much the photo dimensions.
2. Or/and making the photography of the scene (moving the camera or the scene itself) at a shorter distance[260] than the eye saw it.

In both the cases, $MTF_{camera,totall}$[261] can be better than the $MTF_{eye,total}$[116]. To obtain this by strategy at point 2, camera plus an optical instrument (as a magnifier or a microscope[262]) can be used (in this case: $MTF_{camera+instrument,total}$[116]> $MTF_{eye,total}$[116] for one or more frequency). But point 2 is used also when we see directly by the eye through an instrument (magnifier or microscope), to obtain again $MTF_{eye,total+instrument}$[116] > $MTF_{eye,total}$[116]. Otherwise, when the object can not be angularly magnified by methods at points 1 or 2, for example because it is at infinity, we have to use a telescope[263] (in conjunction or not with a camera): in this case its entrance pupil diameter must be greater than the eye one.

---

[258] Up to now we used an *MTF* function of an angular spatial frequency [cycles/°] that is independent from object distances; if, instead, we express the spatial frequency in [cycles/mm in *object space*], we have an *MTF* that is function of object distance.

[259] But not eye information capacity, in particular the spatial information quantity captured by the eye+brain, that is fixed (see the Appendix).

[260] If the scene can be approached (for stars it can not).

[261] We’re expressing here the spatial frequencies in [cycles/mm in object space].

[262] In case of magnifier and of microscope, the perspective rule is not respected, but it could not any more be respected due to a too near *P.o.V.* (see also a photography taken at less than 25 cm = emmetropic near point). Also for the telescope, in a special manner, the perspective rule “decays” because the angular positions of the objects at infinity are no more the same, but in this case depths can not estimated by perspective.

Up to now we spoked about emmetropic eye. Myopic eye, for example, has the same $MTF_{eye,total}$ of the emmetropic eye when measured in cycles/°, but, for an object at myopic near point distance, it has a better $MTF_{eye,total}$ when measured in cycles/mm in object hemispace, for the same reason (angular magnification) of magnifier and microscope.

[...]

Fig. C.1 – MTF independent from object distance and MTF referrred to object distance in [cycles/mm] with amplification

To cheat the eye, it is necessary to use psychophysiology and the parameters that brain uses to judge the reality of an image, then the thresholds on which we can have the "freedom" to decrease performance. Also a bi-objective photo is a 2D representation and not the reality, then viewed by a camera that has a certain distance between objectives that is in general different from the eyes of the person that sees the photography.

The depth of field of the photo must be like the depth of field of the scenario seen by the eye. The two *d.o.f.*s can be the same at far distances, but not at short distances, as seen by the following formula:

…………

Besides, at near distance (macrophotography and close-up) 3D photography is indispensable.

C.2 The "perfect" photography: example for the case of myopia

Because of the goal of preservation of the perspective, the minimum object distance (25 cm) for the *emmetropic* eye is not taken into consideration (and also telescope addition to the camera is not taken into consideration).

In an ironic way, the "perfect" reproduction of the sight, according to a certain parameter[264] is achievable when the eye is not "perfect" regarding that parameter, in particular when the eye is defective (ametropia). In case that the parameter is sharpness, eye has a focusing defect, for example: myopia or hyperopia[265]. In these cases, photos at both far and short distances, respectively, are good because, in general sharpness of the photo can be as good as the defective eye, and:

---

[263] For infinitely distant objects the telephoto lens is not able to improve the resolving power: a telescope (+camera) is necessary.

[264] Here, in particular, the sharpness (the most "evident" in myopia).

[265] So the focusing mechanism is here "normal".

1. For a short distance scene, the camera *d.o.f.* can be enough to emulate the myopic eye (not when the myopic eye focuses nearer than the myopic near point because the $MTF_{myopic\ eye}$ is like an emmetropic eye, so it can not be approached by a photo); 2D photography is a good method because at short distances very likely the near-sighted person closes one eye, due to very short distances in focus respect to the intraocular distance, and his *d.o.f.* is very short.

2. In case of a long distance scene (hyperopia), ...

Assuming that the photo is seen by an emmetropic[266] eye, the condition on sharpness is the following one (see Fig.0 of the dream paragraph):

$$MTF_{camera,\ total} = (Fresnel)MTF_{defective\ eye,\ optics} \,/\, MTF_{normal\ eye,\ optics}$$

C.2.1 The case of the myopic eye

...

C.2.1.2 Calculation of the out-of-focus amount for the myopic eye

To give an idea of the amount of the extra-elongation (delta position of the retina vs. far focal point) of a purely axial myopic eye, the delta δx is equal to … mm for an eye having -2D and …. mm for an eye having -10D (dioptres can refer to contact lenses or eyeglasses lenses or theoretical refractive eye deficiency (in the theoretical case, the rear principal plane of the ophthalmic lens is in contact with the front principal plane of the eye, so the lens is at the Knapp distance)): these delta positions are very short because the rear focal length of the eye is very short (around 20 mm, the eye has a total optical power of around +60 D)

To correct the “poor” vision of the near-sighted person, the eye can not be shortened, so the refractive power of the complete system (axial myopic eye + ophthalmic lens) must be decreased to a normal value. To make this, there are two ways:

1. The refractive power of the cornea is decreased, generally by laser (LASIK, Radial Keratotomy, etc.)

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[266] A “true” emmetropic eye or a corrected ametropic eye.

2. A lens[267] is added, in contact to the cornea (contact lens) (see the following C.2.1.2.1) or at some centimetres in front of the eye (eyeglasses) (see Par. C.2.1.2.2). In the first case, due to the smaller transversal dimensions of the lens, also its central thickness decreases, so the contact lens goes near to the "thin lens" model, gaining in transmittance, weight, refractive absolute power, side thickness, etc..

From one of the prescriptions (for contact lens or eyeglasses[268]) of your ophthalmologist, you can calculate the out-of-focus of your eye[269], then the required out-of-focus for the camera objective.

C.2.1.2.1 Calculation for the contact lens case

To calculate the back principal plane of the contact lens, we must know the first radius of curvature of the contact lens, the second radius of curvature being around equal to the first radius of curvature of your cornea[270], because the contact lens must be in contact with the cornea. Then $r_{2contact\ lens} = \approx r_{1cornea} = \approx 0.0078$ m (= 7.8 mm). Another element that we know, is the power of the contact lens (in air). We don't know the (central) thickness of the contact lens, but touching it, a thickness of about 0.4 mm can be estimated; the refracting index of the contact lens can be supposed equal to N.NN (PMMA) for a rigid gas permeable contact lens. Now, from: P, $r_{2contact\ lens}$ and $t_{contact\ lens}$, we can calculate $r_{1contact\ lens}$, in other words, the power P_1 of the first surface of the contact lens:

$P = - D / (n_{contact\ lens}\, r_1\, r_2)$ then $r_2/D = -1 / (P\, n_{contact\ lens}\, r_1) =$ (for $P = -9$ D)

$r_{1\ contact\ lens} =$

so we know the parameters of the first order model for the contact lens:

d' = ………

We know that the eye can be at a first level approximated using its the cardinal points, so, from the formulas of two-lens combination (APP. 2.2) we can write, from $P_{eye,\ uncommodated} = +59.945$ D, and using the APP.... on the eye, we can calculate[271] the $\Delta s' = \ldots$ mm.

---

[267] the "ophthalmic" lens

[268] See: A. Paolini, *L'incidente stradale*, Mathforlife.net Project (https://archive.org/details/IndaginiDiUnRicercatoreprimoEpisodio) for the limits of the prescription for eyeglasses.

[269] The extra-elongation only of your eye if you are affected by a pure axial myopia.

[270] Measured by corneal topography.

[271] See *A.2.1.1.1.1*, the sub-paragraph dedicated to the ametropic case.

C.2.1.2.2 Calculation for the eyeglasses lens case

......δx……… → circle of less confusion =....

Note that contact lens, being nearer to the front principal plane of the eye, has a lower power than the "equivalent" eyeglasses lens. Another observation is that, in the case of eyeglasses lens[272], the image height on the retina is lower[273] (lateral magnification…..) (so with a worse resolution by the fovea, but the *f.o.v.* is increased) (then the result depends if the photography is seen by an emmetropic or by a corrected eye).

Now, if the $\delta x(s)$ is known, we can calculate the defocusing $\Delta s'$ of a single object distant s from the eye (from the front principal plane of the eye), always neglecting the aberrations and the diffraction effects.

…………

C.2.1.2 Equivalent defocusing calculation for the camera

We would like that, seeing a photography from a certain distance, on the retina the same circle of confusion is created as if we would see the real object. So we would like to have on the retina the same "circle" calculated above..

We can distinguish two situations: when the scene is 2D or when it is 3D.

C.2.1.2.1 2D scene

In this simple case, the objects are on a plane or their relative longitudinal distances are negligible compared to the *d.o.f.* of the camera objective at that distance(s) (this is the case, for example, of objects very far from the objective, for example landscapes). In this case, all objects have around the same distance, so...

C.2.1.2.1 3D scene

For example in near shots (close-up and macro-photography) you have always 3D scenarios. Near in-focus objects are minimum TOT centimetres from the front principal plane of the camera objective, then as in the myopic eye.

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[272] See: A. Paolini, *L'incidente stradale*, (https://archive.org/details/IndaginiDiUnRicercatoreprimoEpisodio) cit., for more details and its implications on the field of view.

[273] it is like the emmetropic eye if, in the pure axial myopia, the Knapp condition is respected, in any case lower than in the a contact lens case.

$x_{obj,nearest}$ = … → circle of confusion for the eye = ….
$x_{obj,nearest}$ = … → circle of confusion for the photo = ….

$x_{obj,farest}$ = … → circle of confusion for the eye = ….
$x_{obj,farest}$ = … → circle of confusion for the photo = ….

Warning: Performance = optical quality (aberrations then in general *MTF*, distortion, etc)

APPENDIX: PRINCIPLES OF OPTICS, PERSPECTIVE AND OF THE THEORY OF INFORMATION

APP.1 The laws of optics (to describe eye and camera) and the perfect imaging

Geometrical optics is geometry, i.e. mathematics, so it is an idealisation (but also an "approximation") of the physical reality. Perfect imaging is in the realm of geometry.

Wave optics is physics, so a better foreseen of "reality".

APP.2 Ideal behaviour in geometrical optics and paraxial (gaussian) geometrical optics

In a first approximation (first order optics or paraxial optics, so without aberrations), all imaging optical systems actuate, magically[274], a projective transformation[275] [276], in general defined by the positions of two focal planes[277], two nodal[278] points and by the two focal “lengths”[279] that are, in general, two tensors[280].

---

[274] Projection without optics (*camera obscura*) can be nearer (except for diffraction) to the geometrical projection concept, but only if *f#* is very high, so with a very little hole (pinhole cameras, with a consequent low luminosity and bad diffraction limit) or, for resolving the diffraction limit, with a large hole but with an enormous camera depth (*camera obscura*), large as a room (see the scale property): it becomes the reference for sharpness in order to see, on the projection screen, object points as *points*, but with very low luminousity (there is not optics that converges the light). So, at a deeper analysis, the magic of optics is not real magic, but, in the spatial visual reproduction domain of photography or of the (pure spatial) eye, only the capability (because object luminous points are not simply geometrical points, but emit spherical waves) to compact the *camera obscura*, so the possibility to move the reproduction system to take it with us. Optics permits to pass from Fresnel diffraction to Fraunhofer diffraction, otherwise raised by an enormous *camera obscura*. In making that, more optics converges, and more it introduces refraction errors (more the Fraunhofer diffraction pattern is not raised).

[275] In optics, conjugate points implicitly imply the concept of focusing.

[276] Only in geometry a projection is possible, so points that project into points (so perfect edges, then perfect shadows). The projection can be a "central projection" (in optics, with focal planes, so *f* and *f'*, and nodal points at finite) or a "parallel projection", see shadow theory and orthogonal projections (in optics, with focal planes, so *f* and *f'*, and nodal points, at infinity, see telescopic projection). In general, the projection transforms lines into lines independently from a scale change of the coordinates (a change of object and image dimensions), along a certain direction determined by the nodal points and according to the system parameters (and the position respect to the focal plane). So the transformation is not a function of the point coordinate, but a magnification (it is invariant vs. a multiplication factor of its coordinates), (see homogeneous coordinates) defined in general by two tensors (taking in general the parallel and orthogonal directions respect to the "object" (front) focal plane). For imaging it is necessary to obtain an image similar to the object (so images that are not coded (deformed in a certain manner), but that are directly visible by the eye, at least in the transversal plane), and this can be carried out only along one direction respect to object planes (for the properties of projective transformations). Then this plane is chosen to be the transversal plane when using a centred revolution systems (see here below), (nodal points are on the axis) obtaining (different) scalar magnifications in transversal and longitudinally directions (see the transversal and longitudinal magnifications in the following pages): the image is longitudinally deformed.

Fig. APP-0: Convergent and divergent not-centred systems with not parallel focal planes. P and P' are conjugate points. [Also if in this equation the prime letters refer to points in the opposite direction of the z axis respect to the optical system and no-prime letters refer to... (there is a direction of the z axis that defines a direction), for clarity it will be supposed that at the left (towards the negative direction of the z axis) there is the object (P) and to the right its “image” (P') ..., then p=to??? and p'=ti???.]

If the optical system is, ideally, perfectly centred and of revolution[281], naming z the optical (symmetry) axis, magnification is transversely uniform (further magic![282]): the two focal planes are orthogonal to z-axis[283] and the two focal lengths are scalars[284]: object and image (points) are related by an equation that has different forms that depend on the reference point for the z-coordinate measurement (the more spread forms are reported: Newtonian form and Gaussian form). Given the z-axis, the projection is defined by four[285] [286] (independent) parameters: (F, F', f, f') or (F, F', f, n'/n) or (H, H', f, n'/n) or (H, *hiatus*, f, n'/n), etc., or the four following cardinal points[287] (F, F', H, H') or (F, F', N, N') or (F, F', N, *hiatus*), etc.. Here, as convention, we use the same signs for f and $f\,'$ for dioptric systems, so Z and Z' have the same sign when go in opposite directions, here positive[288] when they go away from the foci. So we can calculate the two equations for Z and radial coordinates[289]: the "longitudinal conjugate points" equation (Newtonian or Gaussian form) and the transversal magnification equation (Newtonian or Gaussian form). We start with the "longitudinal conjugate points" equation, in the *Newtonian form*:

---

[277] In general not orthogonal between them.

[278] As projection points, the nodal points are valid only in paraxial approximation where the pencils of rays are surely centred, or near, around the nodal points (see Note 161). In real cases, where the pencils of rays, in general, can pass decentred from nodal points due to the pupils positions not coincident with nodal points (if Pupil Magnification is different from 1 (“asymmetric” optical system in aperture)), the projection points are the centres of pupils; in any case, off-axis aberrations bring the system outside the ideal projection case.

[279] They determine the “power”, P, (the strength) of the projection ($P = n/f = n'/f'$). They can be positive or negative and with their combinations of sign they determine if the system is katoptric or dioptric and if it is convergent or divergent.

[280] The matrix (tensor), can be expressed choosing, for example, two orthogonal meridional planes (two planes orthogonal to the focal plane and orthogonal to each other).

[281] It is true if curvatures of spherical or aspherical surfaces AND indexes of refraction have a rotational symmetry around the optical axis. So here we are excluding the unavoidable defect named astigmatism (see Note 68), that brings back to the general case. Thanks to this symmetry, the tensor can be diagonalised using eigenvalues with the two diagonal focal lengths equal).

[282] As it will be seen in the following, simply spherical reflecting and refracting surfaces will be used to give this “unbelievable” result.

$$Z \cdot Z' = f \cdot f\,' \qquad (1)$$

that can be usefully written, using the "normalized"[290] Z-coordinates, Z'/f' and Z/f, in the following form:

$$[Z'/f']/[Z/f] = 1 \qquad (1\text{bis})$$

When the distances, s and s', are taken from the Principal Points instead that from the foci ($s = Z + f$; $s' = Z' + f\,'$)[291] the "longitudinal conjugate points" equation takes the *Gaussian form*:

$$\frac{n}{s} + \frac{n'}{s'} = \frac{n}{f} = \frac{n'}{f'} = P \qquad (1\text{ ter})$$

---

[283] For this revolution symmetrical case, in general the two focal planes have an inter-distance equal to $f+f'$ plus an additional hiatus. This hiatus doesn't enter into longitudinal and transversal magnification equations, but it is related to longitudinal separation between object and image (virtual image in dioptric systems can be before or after the object depending on hiatus, see later).

[284] Instead of the two (scalar) focal lengths we can use: the (scalar) optical power of the system (calculated for $n' = n$) and the ratio n'/n (so f ed f' contain inside the information of n ed n').

[285] In this case, the four parameters for each half-space of the general projection (the plane inclination in space and the tensor 2x2 of the focal lengths) are reduced to two for each *half-plane* (someone of the meridional planes).

[286] Further simplifications/particular-cases will be analysed in the following, where only three parameters are necessary: $n = n'$ (same media in object and image half-spaces) $\rightarrow f = f'$ and $N \equiv N'$; $H \equiv H' \rightarrow H \equiv H'$ [→ hiatus = 0]: thin lens model.

[287] The six cardinal points and the four cardinal planes have useful properties that can be used in geometrical-graphical construction of conjugate pairs in paraxial regime. We use two conjugate pairs of cardinal points(/planes), well chosen because not all the six cardinal points are independent: if $f \neq f'$ (if N and N' do not coincide with H and H') H, H' are not independent from both N, N' (N and N' are shifted respect to H and H' of the same quantity ($f' - f$) (see Fig. *APP-1* for the sign): the distance between them, the hiatus, is the same), so a geometrical-graphical construction using these four points doesn't exist, see what follows.

[288] With a symmetry between "object" and "image" half-spaces (we are not using the same versus of the z-axis, but different directions in each half-space): each point is conjugate to the other.

[289] These equations are valid for any type (real or virtual) of object or image point, see Fig. *APP-4*.

[290] putting in light the scalable nature of equations of a projective transformation.

[291] Nodal Points could be other references for the z-distances, but generally an equation with these z-references is not used.

where P is the optical power[292] of the system (a unique quantity that characterize the system by itself (its curvatures and its various internal $n_{internal}$), independently from external media (that, in this case, are like they are fixed to $n = n' = 1$) (focal lengths of each single surface are instead two and depend on refraction indexes at left and right of it)).

It can be usefully written, using the "normalized" s-coordinates, s'/f' and s/f , in the following form:

$[s'/f'] = [s/f] / ([s/f] -1)$ (1 quater)

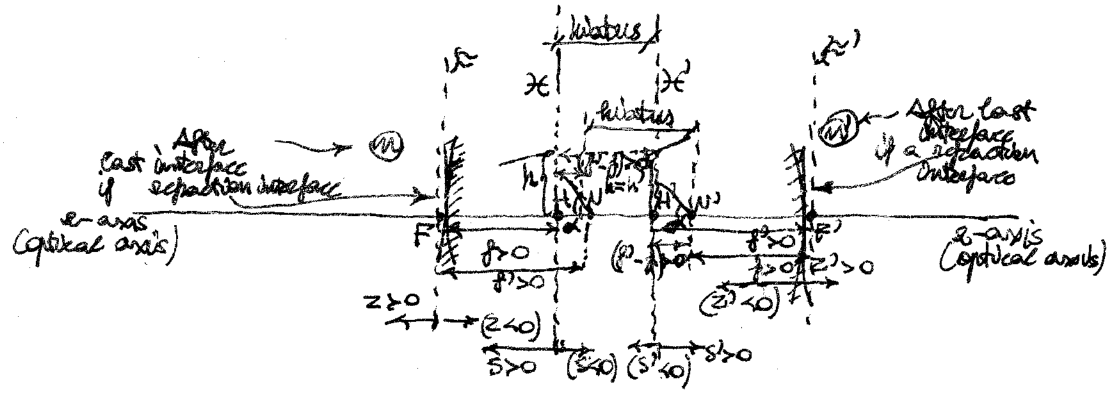

Fig. APP-1: General scheme for all dioptric and katoptric (and katadioptric) centred revolution systems. Here |f'| > |f|.

With anyone of the two formulæ (Newtonian or Gaussian one), we can calculate the position of the other conjugate point along the optical axis, to complete the pair ((Z, Z') or (s, s')).

---

[292] All together the internal characteristics of the optical system determine only one parameter (the optical power): according to P, the behaviour would be independent from the ray direction (internal dielectrics are not dependent from z direction)): in reality there are two projective "strengths" parameters f and f , the difference between them is determined by the external means (n and n').

For the transversal distance[293] of the conjugate point, we need to calculate the (linear)[294] transversal magnification[295] [296] (the transversal[297] magnification is supposed to be constant[298] on all the transversal sections of the object half-space), that depends only by one of the conjugate points[299]:

$$M_{transv}|_{z\text{-}coordinate=constant} = \frac{r'}{r} = \frac{-f}{Z} = \frac{-Z'}{f'} = \frac{-Z'}{fn'/n} \qquad (2)$$

$$M_{transv} = \frac{-f}{s-f} = \frac{-(s'-f')}{f'} = \frac{-s'}{f'} + 1 \qquad (2\ \text{bis})$$

(according, in the order, to the Newtonian or to the Gaussian (normalised) z-measure).

In particular, the first term of (2) and (2bis) is useful to calculate the magnification from the position of the object and from f. Points on Unit (or Principal) conjugate planes H and H' are conjugate points (according to the Newtonian or Gaussian forms, and also according to M_{transv} formulæ[300]). This property will be used in geometrical-graphical constructions (see below).

The *angular* [transversal] magnification, $M_{ang}|_{z\text{-}coordinate=constant} = \tan\alpha' / \tan\alpha$, is related to the linear transversal magnification[301], as will be useful in *f.o.v.* calculation (there M_{transv} will be substituted by P_M):

$$M_{ang}|_{z\text{-}coordinate=constant} \cdot M_{transv}|_{z\text{-}coordinate=constant} = f / f' = n / n'$$

We can derive, in particular, the position of the nodal points N and N' that are the conjugate points where $M_{ang} = 1$: using (2bis), $s_N = f - f'$ and $s_{N'} = f' - f$.

From the Gaussian first or second expression (1ter or 1quater), substituting the (1) we draw the "nodal point" form for M_{transv}:

---

[293] The transversal distance to the axis (using a radial coordinate with *sign*) is a linear dimension and not an angular one: the angular extension (that links transversal magnification to z-position) is instead constant if measured from certain z-points (see later in Appendix): the perspective is maintained, but if there is not an "inversion" of the distances, as in the magnifier configuration (and also microscope), see further).

[294] For distant objects, to obtain a high linear magnification we can increase f (so f'), but for very distant objects (at infinite) the only possibility is to use a telescopic system (convergent (erect image) or divergent (inversed image) one) that gives an *angular* magnification greater than 1, loosing perspective (the non-telescopic systems give an *angular* magnification equal to 1).

[295] A projective transformation brings to a magnification that doesn't depend on the r value, but on z-axis coordinate.

[296] This theory applies to all α angles (so to tangents of angles): $M_{transv}|_{z\text{-}coordinate=constant} = dr'/dr = r'/r$, but in reality it is used, approximated, only for small angles (where $\tan\alpha$ =around α).

[297] respect to the optical axis; the same for the longitudinal magnification.

[298] it is constant if the system has, globally, a symmetry respect to the optical axis (a rotational symmetry that can guarantee a non-astigmatic system, at least paraxially around the axis).

[299] Given M_{transv}, knowing f and f', we can calculate s and s' (see, for example, in the following, pupil positions calculation from P_M).

[300] The image, at the same radial distance, is virtual.

[301] A conjugate ray passes through two conjugate points, so, as it is convenient to choose: on-axis points are one pair of conjugate points, the two points at the same z-axis distance, on principal planes, are the other conjugate pair.

$$M_{transv} = \frac{-\frac{s'}{n'}}{\frac{s}{n}} \qquad \text{(2 ter)}$$

that confirms that, if Nodal Points coincide with Principal Points (when $n = n'$), if we see from the first Nodal Point the object and from the second Nodal Point the image, the subtended angle is: $r'/s' = r/s$[302]. If the image generated by an optical system in concurrent systems is on the opposite side of the object[303], as in the eye and in the camera, the perspective is maintained if the *P.o.V.* points are respected . As it is clear from the longitudinal equation, *F* and *F'* are the points that......

We can use both Newtonian longitudinal equation (1bis) and M_{transv} with normalized z-coordinates to create a graph:

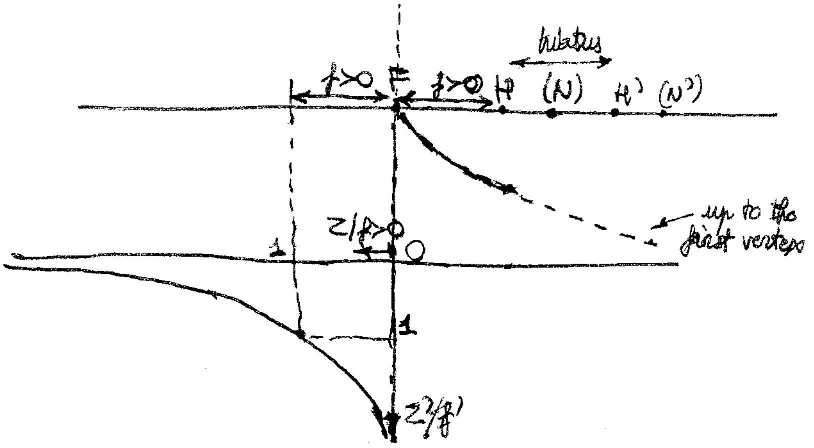

Fig. APP-2: Transversal magnification (changed of sign) using "normalized" Z-coordinates for a dioptric convergent revolution system.

And the same using Gaussian ones:

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[302] See Note 147.

[303] A magnifier doesn't give an image according to perspective, see later.

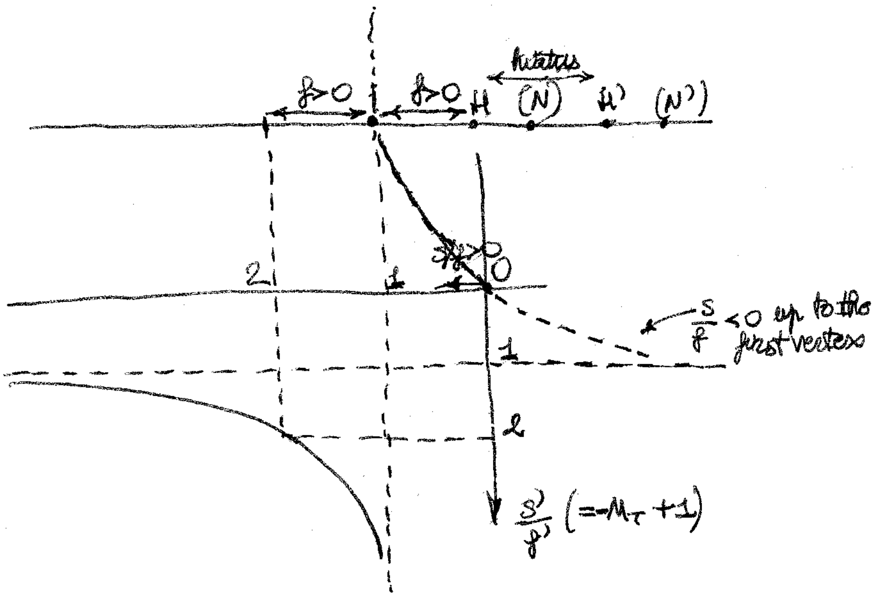

Fig. APP-3: "Longitudinal conjugate points" Gaussian equation using transversal magnification and "normalized" longitudinal coordinates for a dioptric convergent[304] revolution system.

The longitudinal magnification[305] is not an independent quantity respect to the two previous equations[306]: it is drawn by one of them, differentiating the "longitudinal conjugate points" equation, but also squaring the transversal magnification formula[307]. We start by differentiating the longitudinal equation:

$$M_{long} = \frac{dz'}{dz}$$

according to the Newtonian z-measure:

$$M_{long} = \frac{-ff'}{Z^2} = \frac{-Z'^2}{ff'}$$

or to the Gaussian z-measure[308]:

---

[304] In a divergent system high magnifications are obtained, again, around f, but in this case with $s < 0$ and not $s > 0$. In the microscope, in particular, s is slightly at the right of f.

[305] It is defined by infinitesimals because it changes with z-coordinate change: $M_{long} \neq Z'/Z$.

[306] In other words, two independent formulæ are enough (because we have to define the two coordinates of the point): [longitudinal equation, M_{transv}] or [longitudinal equation, M_{long}] or [M_{transv}, M_{long}].

[307] M_{long} is independent from r and r', confirming that longitudinal equation and M_{transv} equation are uncoupled formulæ.

[308] We can differentiate the Gaussian "longitudinal conjugate points" equation or we can also substitute $Z = s + f$ and $Z' = s' + f'$ in the previous formula.

$$M_{long} = \frac{-f\,f'}{(s-f)^2} = \frac{-(s'-f')^2}{f\,f'}$$

Its relation with M_{transv} is:

$$M_{long} = \frac{-f'}{f} \quad M^2_{transv} = \frac{-n'}{n} \quad M^2_{transv}$$

M_{long} is always < 0 for a dioptric (concurrent) centred system, so also for all-refracting surfaces systems that are considered here.

M_{long} is always different from M_{transv}, except when $M_{transv} = M_{long} = 1$, so practically the 3D image is always deformed along axis respect to the original 3D object.

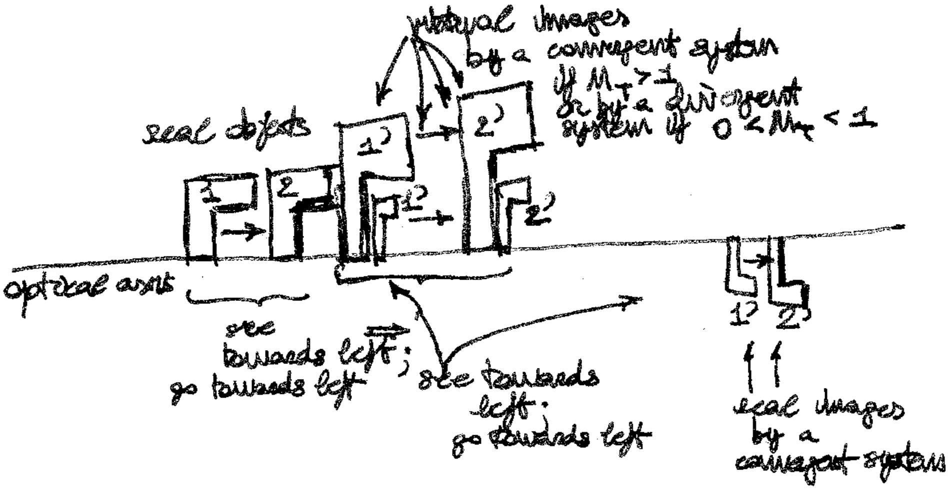
virtual images by a convergent system if $M_T > 1$ or by a divergent system if $0 < M_T < 1$
real objects
1
2
1'
2'
1'
2'
optical axis
see towards left; go towards left
see towards left; go towards left
1'
2'
real images by a convergent system

Fig. APP-4: Longitudinal magnification consequences on versus and displacements of the images vs. object ones, for dioptric systems, here in particular for a centred revolution system.

We can summarize the longitudinal and transversal equations for a dioptric system (focal lengths of the same sign), in the following picture:

[Picture with Ls]

Fig. APP-5: (a): for constant and positive f, f' (as camera with fixed focal length) [→ perspective rule is NOT verified], (b): positive and variable f, f' [we suppose to vary F to focus → perspective rule on the subtended angle (at the eye) is clear], (c): negative and constant f, f'. According to the sign of f, f' , after the systems will be named convergent and divergent ones). In the only case of Mt=1 and Mlong =1 there is no longitudinal deformation .

To find the conjugate[309] of a point by a geometrical-graphical[310] construction, we can use one of the following two methods that use the properties of the cardinal points and planes[311]. We have to use two conjugate pairs of cardinal points(/planes) that are independent[312] (because we have to define a point in the half-plane (two lines), we need two parameters, and other two conjugate parameters in the other half-plane for the other two lines), but we need always H and H' to define the radial distance coordinate[313].

---

[309] You can start from a point or from the other: the “starting” point (without apostrophe) is named “object”, the “arriving” point (with apostrophe) is named “image” (this is due to the application of the causality constraint).

[310] Geometry is graphics and viceversa.

[311] Considering properties and positions of them, it is simple to derive both the “longitudinal conjugate points” equation and the M_{transv} equation from one or from the other of these geometrical-graphical constructions (by triangle similitude) (in other words, geometrical-graphical construction is the real objective of geometrical optics, it is the optical geometrical equivalent of the projective transformation).

[312] See Note 268.

[313] See the previous discussion: one point on one Principal plane is imaged (virtual image) on the other conjugate Principal plane at the same radial distance, so each ray that arrives on it with a certain inclination can be shifted of the hiatus quantity along the optical axis, so at the same height, onto the other Principal plane (with its sorting inclination in general undefined, except for the case of horizontal ray that passes through F'). Only points on Principal planes have to be taken into consideration, not the space in the hiatus that can also have optical power: internal path is not horizontal and not known using cardinal points/planes (and not interesting for the construction).

Using the conjugate pair (*F*, *F'*)[314] and the two conjugate planes *H* and *H'*:

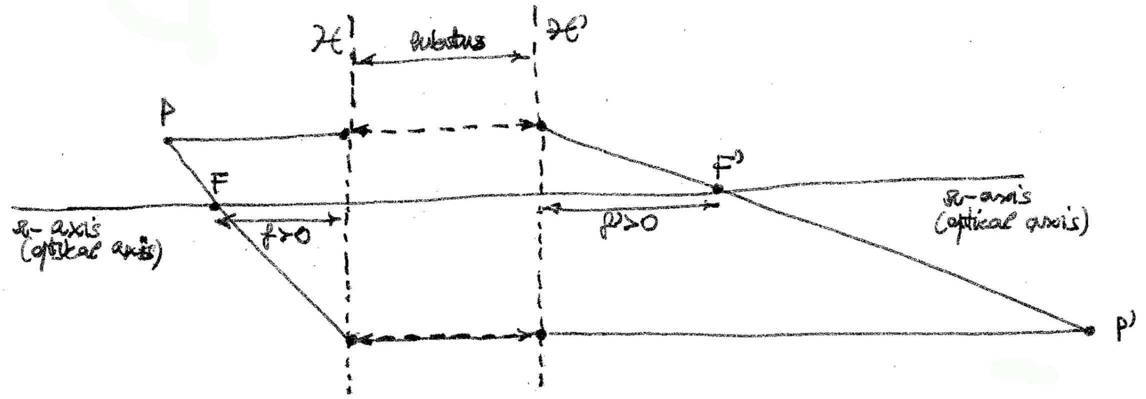

Fig. APP-6: The method is general for a non-telescopic system. Here the convergent and divergent (To Be Added) cases for a dioptric system. The optical system is the same of Fig. APP-7 below, so the conjugate pair (P, P') is in the same position. OPTICAL SURFACES (two or more) To Be Added (To Be Corrected: f and f' are valid if ALL the surfaces are passed through)

Using one between *F* and *F'*, the two conjugate planes *H* and *H'* and the conjugate pair (*N*, *N'*)[315]:

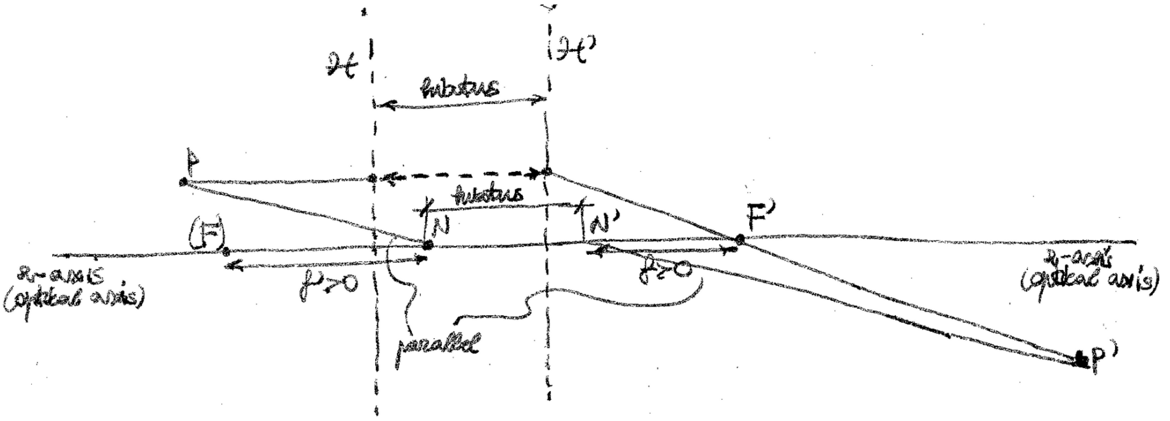

Fig. APP-7: The method is general for a non-telescopic system. Here the convergent and divergent (To Be Added) cases for a dioptric system. The optical system is the same of Fig. APP-6 above, so

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[314] Only rays that go or come from *F* and *F'* are/must-be horizontal.

[315] This method substitutes *F* or *F'* by (*N*, *N'*). Remember that nodal points are valid as projection points only in paraxial approximation (like also the validity of the previous construction) where the pencils of rays are surely centred, or nearly centred, around the nodal points (see Note 123).

conjugate pair (P, P') is in the same position. Here F' is (explicitly) used, but, alternatively, F can be used. OPTICAL SURFACES (two or more) To Be Added (To Be Corrected: f and f' are valid if ALL the surfaces are passed through)

Fig. APP-8bis: Geometrical determination of size and position of entrance from the exit pupil or viceversa. The method is general for a non-telescopic system and a telescopic system. We need two non-parallel rays

APP.2.1 The type of optics for imprinting reality

As explained in Par. A.1.1, it is not important if the system is a dioptric or a katoptric one (or a katadioptric one), but it is important to have a system that gives a real image (so a convergent system): to use reflecting elements (in photography, because in the eye we have only refracting elements) we need to have a not centred system or a centred system that doesn't use all the aperture (→ low aperture efficiency). Here we will treat only dioptric (concurrent) centred systems (with the adopted convention: $M_{long} < 0$)[316].

---

[316] Note that surfaces that constitute the dioptric system, constitute an undesired katadioptric system due to their Fresnel reflections at interface (the surface in front of the object gives the major contribution), so centred katoptric systems are not treated here but in any case are always useful to calculate power loss (with pupils and stops information, and considering Fresnel reflection value for each incident direction) and "ghost" images of parts of the photo*receiver* itself (the receiver can not be at the same time transmitter) (see: A. Paolini, www.lartedelsorriso.eu) (consider also reflections of part of the retina that can be seen by his own nuke eye with strong lateral illumination and dark ambient illumination, but also concave mirror used as a magnifier) or of internal part of optical system (illumination must be not orthogonal to the reflecting surface and background light smaller). Considering only "ghost images", photographic objectives and nuke eye are designed to minimize them; instead eyeglasses (that never have a wraparound frames) are not (see: A. Paolini, *cit.*) (instead contact lenses yes).

Centred katoptric system constituted by the refracting surfaces can be also used to explain what we can see if we see the optical (kata)dioptric system from outside, see Note 370 for the beautiful image seen in the pupil of the eye.

For a dioptric convergent system (*f* and *f'* >0) the general Figure APP-1 can be taken because drawn in a favorable manner:

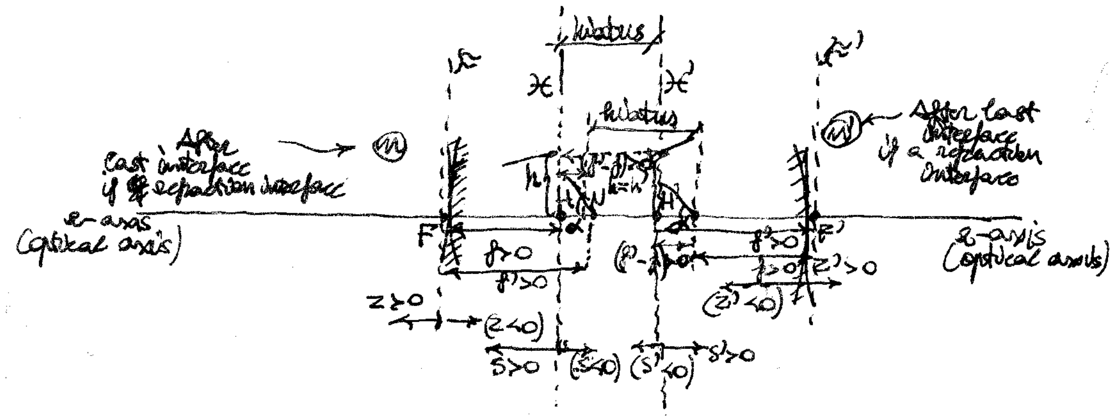

Fig. APP-8: Scheme for a convergent dioptric (f' and f > 0) centred revolution system.

For a dioptric divergent system (*f* and *f'* <0), we can customize the above picture with *F*, *F'*, *N* and *N'* tipped over respect to *H* and *H'*:

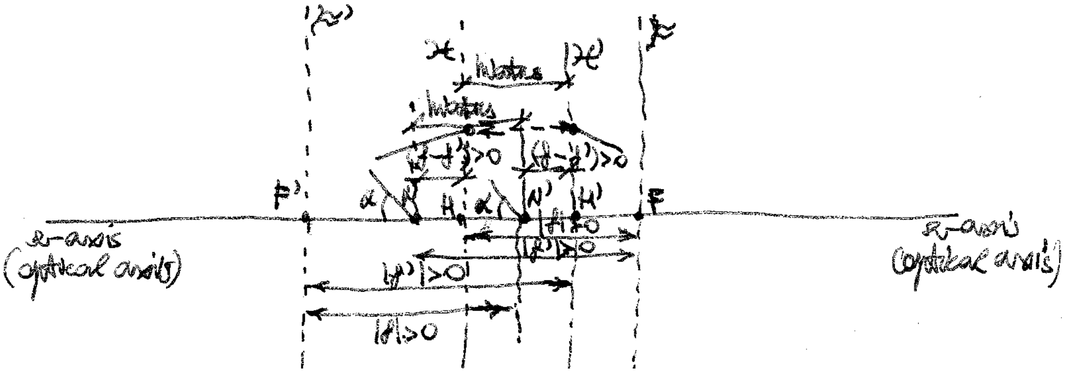

Fig. APP-9: Scheme for a divergent dioptric (f' and f < 0) centred revolution system.

The type of points and the perspective rule

Both image and object points are classified as real or virtual. Referring to the below figure, virtual object is not interesting for a complete system (for which the object is always real[317]). A point is classified as "object" or "image" according to a temporal order (cause-effect) that gives to light rays a direction[318] that starts from the real or from the virtual object (both emit and "are not seen") and arrive to an image point (real or virtual) (which "is seen[319]").

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[317] A virtual object can be present as an input of an intermediate (not the first) system of a cascade.

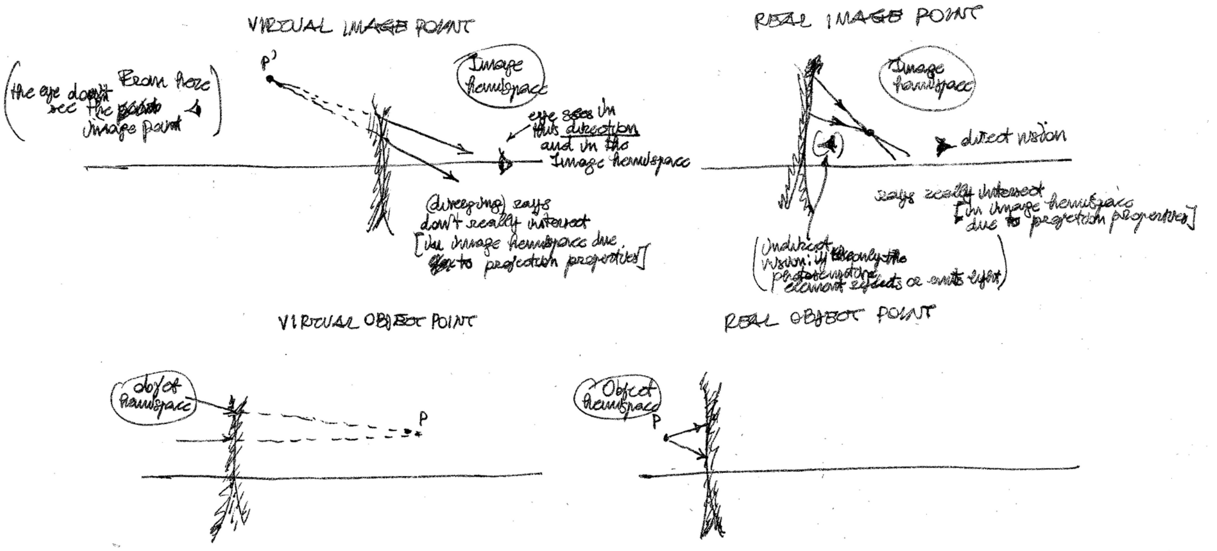
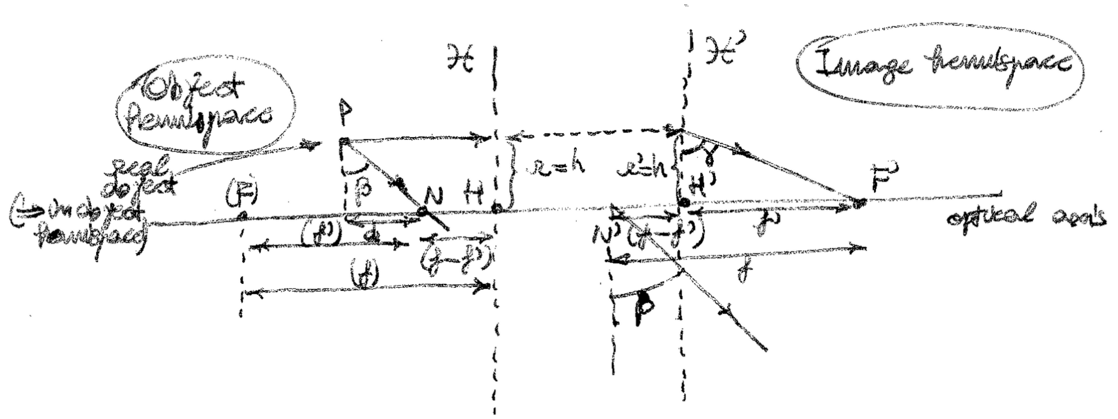

VIRTUAL IMAGE POINT
P'
(the eye doesn't see the image point from here)
Image hemispace
eye sees in this direction and in the Image hemispace
(diverging) rays don't really intersect [in image hemispace due to projection properties]
REAL IMAGE POINT
Image hemispace
direct vision
rays really intersect [in image hemispace due to projection properties]
(indirect vision: if the photosensitive element reflects or emits light)
VIRTUAL OBJECT POINT
object hemispace
P
REAL OBJECT POINT
Object hemispace
P

Fig. APP-10: Definition of real and virtual categories for object and image points.

On the contrary, for our intents the image classification is very important, because the photosensitive element can not reflect/emit/memorize each *single* ray that is incident on it (each from a different point!), so the optics must form on the photosensitive element a *real* image point[320]. Then we have to understand when we obtain a real image using a dioptric system:



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[318] This is not relayed to object and image half-spaces that can coincide (katoptric systems) or are at opposite sides (dioptric systems).

[319] Note, in the picture APP-10, the eye position and versus in the two cases of image and no eye near the two types of object.

[320] The object point is surely real because here we are considering the total system.

Fig. APP-11: calculation when the image is real or virtual in a dioptric (convergent or divergent) system: because the system is dioptric, object and image half-spaces are at opposite sides. We have chosen object rays coming from left, so the object half-space on the left. We are considering a global system, so object point is real (→ in the object half-space). We use the geometrical-graphical construction with (N, N') (we use N,N', H, H', F').

Starting from a real object:

$$\beta = \tan^{-1}\frac{d}{h}$$

$$\gamma = \tan^{-1}\frac{f'}{h}$$

The conditions to have a real image are:

1. With *N'* before *F'* (convergent system ($f, f', > 0$)):
b > g → $d > f'$ then the object to the left of *F* (instead if d < 0 → b < 0 always, so image is always virtual)

2. With *N'* after *F'* (divergent system ($f, f', < 0$)):
g > 90° and P always at the left of *N'*
→ $b > g$ never verified, then the image is always virtual

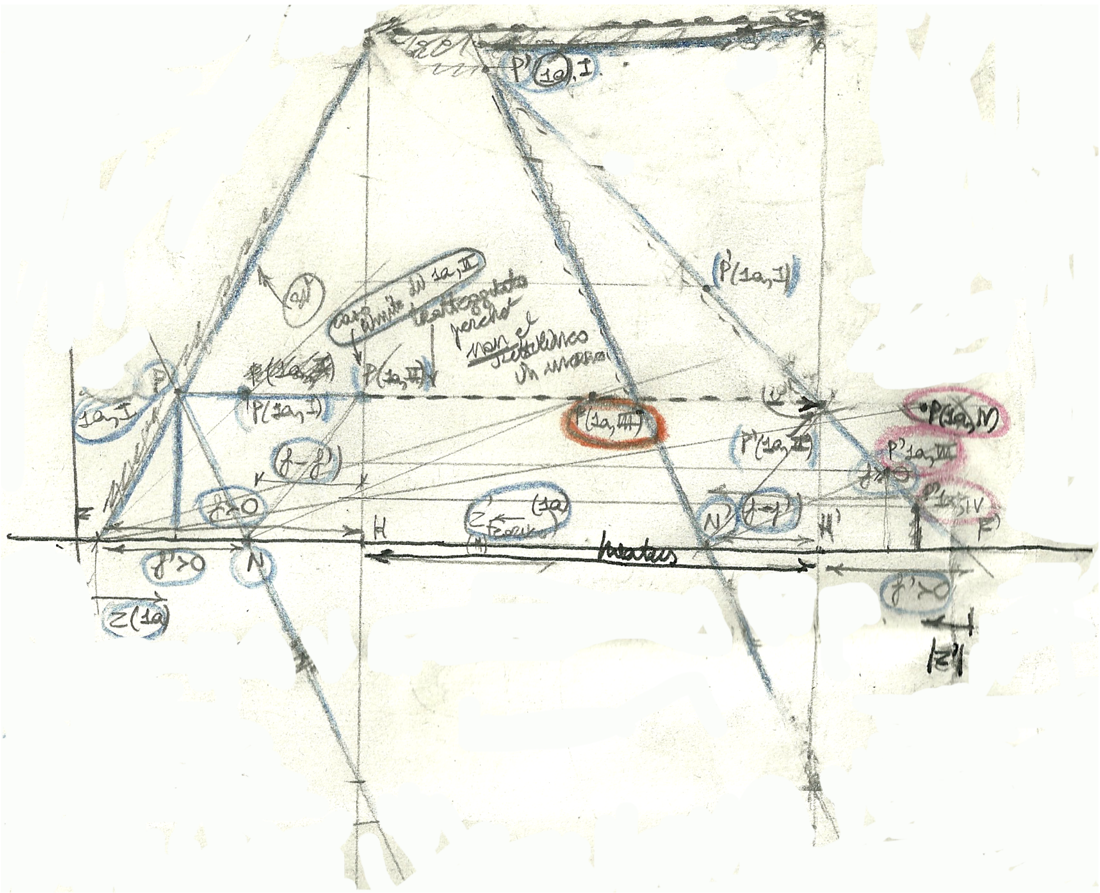

Fig. APP-12: A study of the conjugate points for a convergent system when object or image is between foci: in this case, independently from the relative position of the nodal points (the sign of (f – f'))[321]*, both of them are between foci. A real object gives a virtual image and a virtual object gives a real image, with M_{transv} that is always positive and that can be also < 1 when the object*[322] *is in the hiatus or beyond it*[323] *(→ s <0) (cases 1a, III and 1a, IV). If hiatus >> f, f', as here, virtual images from real objects, for not enormous M_{transv} , are nearer than object. OPTICAL SURFACES (two or more) To Be Added (To Be Corrected: f and f' are valid if ALL the surfaces are passed through)* so the dioptric system must be convergent[324] [325]AND with the object before (to the left) the object principal focus *F*.

---

[321] In the picture the case *f* > *f'* is depicted.

[322] Changing *Z* ← → *Z'* (or *s* ← → *s'*) and *f* ← → *f'* we can change object ← → image, but for M_{transv} we have to precise which is the object and the image, because substituting object to image and vice-versa M_{transv} inverts.

[323] To be in the hiatus or beyond it (but also before the hiatus) could be physically impossible for a real object, because of the presence of the impenetrable dielectric of the lens(es): in this case, a virtual object can be projected there by another optical system. Nevertheless, both in single (thick) lens and in complex systems the Principal planes can be at the left of the single lens or of *all* the lenses, so a real object can be putted at the right of *H* (*s* <0). In the divergent system case, this position of the real object is used in microscope.

For a dioptric convergent system, taking in mind Figs. *APP-2* and *APP-3*, we can tray the graph of the transversal magnification equation[326]:

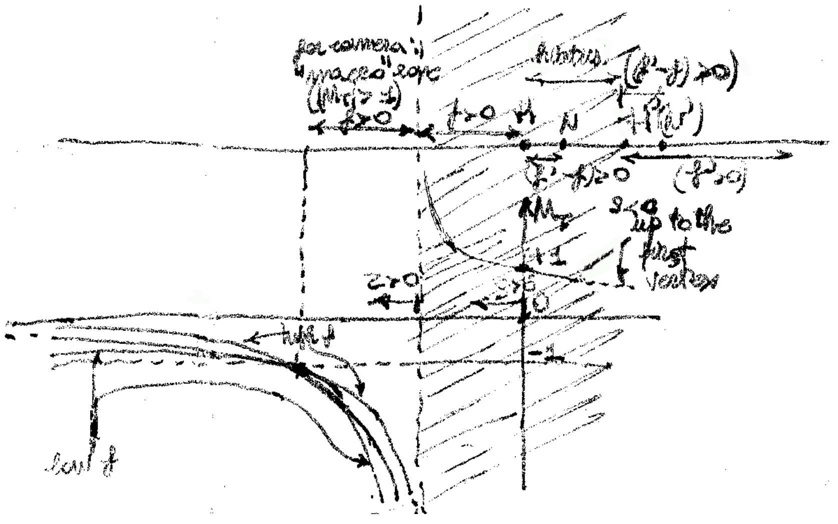

Fig. APP-13: Transversal magnification equation for a dioptric convergent revolution system, with shaded the "interdicted" zone (for camera) of not real images. "Macro zone" (macro regime) gives magnification: in case of the camera, the image is real and the object focus is approached from ahead, in case of the magnifier, instead, the image is virtual and approached from the opposite side.

APP.2.1.1 The focusing mechanisms and their ranges; the perspective rule.

The not-shaded focusing zone in object semispace in Fig.*APP-13* must be reduced in practical cases because the photosensitive element is placed at a certain finite distance from *F'* (or from *H'*), so the object can not be focused if it is too near to *F*. Starting from objects at infinity, that are simply focused on the finite second focal plane, the goal is to focus objects as near as possible (the theoretical limit continuing to be Z = 0 or $s = f$). Considering the geometrical scaling property[327], the first consideration is that decreasing (f, f'), the dimensions of the complex optical system (photosensitive element comprised) are proportionally decreased[328].

First of all, it is necessary to analyse the focusing strategies. Considering the case of a fixed or quasi-fixed[329] photosensitive element respect to the objects, we can play on the four available independent parameters, also if, with n and n' fixed, really we have only three parameters to use: (f, H, H'); so we can classify the focusing mechanisms, for example, in this manner:

---

[324] Dioptric divergent systems give always virtual images (minified, but also magnified, see the microscope).

[325] The convergent system can be formed also by negative elements, but with total equivalent f' and $f > 0$.

[326] For the zone with $s < f$, see also the previous figure.

[327] Except for the hiatus, but it is proportional to t, see the following note.

[328] If in a complex system composed by multiple lenses, all (f, f', distances (the *t*s) between lenses) are multiplied by a factor, dimensions (hiatus included!) of the entire system are multiplied by the same factor.

- Using fixed focal length(s) objective (as possible in the camera): in this case we have three choices: to decrease the width of the hiatus[330], or to move away it from the photosensitive surface[331] [332] or to move away the entire[333] optical system respect to the photosensitive surface. As a consequence, the camera + objective length increases when near objects are focused.

- Decreasing the global focal length (as in the eye), normally by changing curvatures and not refraction indexes: width and position of the hiatus[334] can change or not[335], but surely principal or nodal points change of position (because the distances H - N and H' - N' change[336]). An advantage in compactness is present vs. the camera: the length[337] of the eye stays constant when near objects are focused.

For the camera with fixed equivalent focal length, we can use, for focusing, the following equation:

$$M_{transv} = \frac{-f}{s-f} = \frac{-(s'-f')}{f'} = \frac{-s'}{f'} + 1$$

and the previous normalized graph:

---

329 Moving entire system, as in camera, instead of shortening focal lengths, as in the eye, for long focal lengths brings that, to focus very near objects starting from far objects, we are also obliged to move away the camera from the near object to focus it. In the eye it is not necessary (we can stay fixed).

[330] with multiple (thick) lenses, it is possible changing only lens's positions.

[331] It is possible using a single lens but changing its curvatures; with multiple (thick) lenses, it is possible changing only lens's positions.

[332] So changing the positions of principal and nodal points.

[333] It is the most spread strategy in cameras with fixed focal lengths.

[334] with multiple thick lenses, is it possible changing only lens's positions.

[335] In the case of the eye, changing the two curvatures of the crystalline lens and its thickness, three degrees of freedom are used, so of the three parameters (f, H, H') not only f is used, but also H and H' change.

[336] They depend on f: $|H - N| = |H' - N'| = | f' - f | = | f(n' - n)/ n |$

[337] For both, camera and the eye, its length (minimum for camera or fixed for the eye) is determined by the focal length, f or f'.

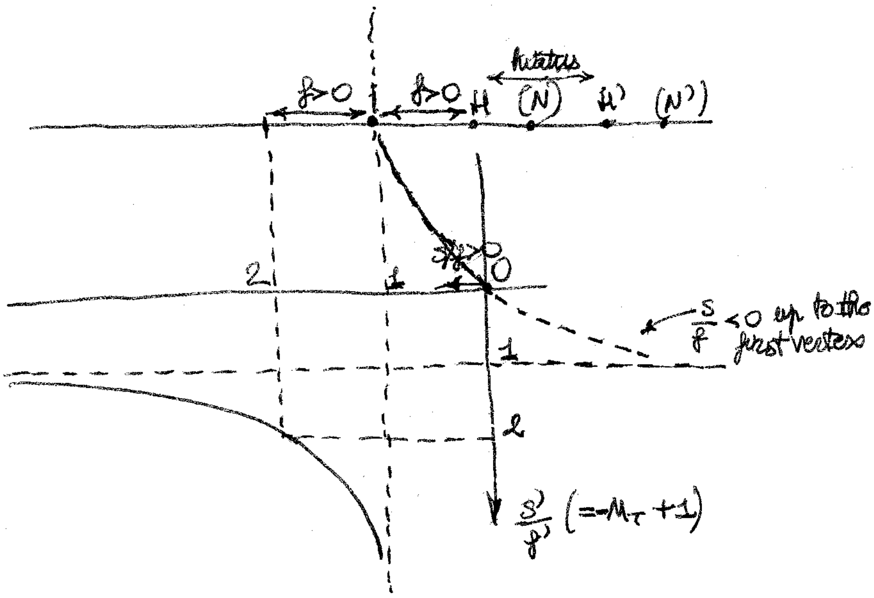

Fig. APP-14: This graph is general, but can be used for camera near object limitation fixing f and f' [To be Dashed the impossible zone]

With $f = f'$, minimum[338] distance of object to be focused ($Z_{min,\ focused}$) is determined by Z'_{max} permitted by the geometry of the camera+objective; in this considerations, all is proportional to f' and f, so for a long focal length the minimum distance and the dimensions of the camera+objective increase. Considering $|M_{long}|$, we have to square the ordinate coordinate of the Fig. *APP-14*: decreasing Z/f, $|M_{long}|$ increases, so the depth of field (*d.o.f.*)[339] decreases.The limit for *d.o.f.* doesn't depend on f if the same M_{transv} is desired (with a short f we have to go shorter to the object to obtain the same M_{transv}, then the same M_{long}).

---

[338] Here the minimum focusing distance is analysed in terms of perfect focusing: the refinement of this analysis, considering the acceptable circle of minimum confusion, according to an "acceptable blur spot", $b'_{blur,accept.}$, for the eye, or an "acceptable blur angle", $\beta'_{blur,\ accept.}$ for the camera (see *A.1.2.1.1.1*) doesn't change the results.

[339] Depth of field depends also on system pupils (f/# and P_M) that can not be considered here where the paraxial case is treated (→ no transversal dimension is considered), see A.1.2.1.1.1.

Fig. APP-15: Useful focusing range for the camera

The practical limitations to the nearest objects are due to: for camera, to the delta-length of (body+)objective and to the total length (see extension tubes and bellows), for the eye, to the maximum curvature of the crystalline lens AND, for both the camera and the eye, $|M_{long}|$ ($1/d.o.f.$) that becomes too high.

Fig. APP-16: Useful focusing range for the eye [M_{transv} To Be Expressed by s' and $s' = f'_{max} = const$ To Be Imposed]

Fig. APP-17: Depth of focus: picture of the manual focusing setting ring that has not distances linearly engraved (also the autofocus moves more at short distances)

We can understand why on the focusing setting ring of a reflex camera a minimum focusing distance (but measured from which? The photosensitive element? Or from the anterior focal point? Or from the first vertex?) greater than focal distance is marked.

First of all, we can see that this distance changes if we use different objectives: for example, in a pocket camera (due to its reduced depth) this minimum distance for a 35 mm objective is about 1meter, whereas in a reflex camera with the same *f* it will be shorter[340]. A reflex camera with a standard objective of focal length of 50 mm is about 10 times the focal length, then 50 cm. If we change objective, we can observe that in objectives with a shorter focal lengths (called wide-angle lenses) the minimum distance decreases respect to a “standard” 50 mm, being always around ten times the focal length (for a 28 mm) around 30 cm), whereas in long focal length objectives, e.g. tele-objectives, this minimum distance increases (for example, ten times 135 mm is equal to 1.35 meters).

In objectives called “macro”, this minimum distance is reduced (not with deeper camera body, but by objective, fixed more distant or with second focal point more ahead), so if you see the focus setting ring of a 50 mm macro the minimum, the minimum distance is …. and for a 135 mm macro the minimum distance is …. Macro objectives exist from 50 mm above, because:

1. Shorter focal lengths are already a “macro”
2. Too short focus distances are too short respect to near point of emmetropic eye
3. They don’t want to permit a near approach to the subject??, because if the object??? is approachable?? at near, it is approachable also farther. Macro objectives are constructed to magnify (or, better, to reduce the mignification) of the object on the film (or the photosensitive element) (see par. A.2.6.3 on transversal magnification).

For small objects macro (generally they have not a bellows, and have $M_{\mathrm{transv}}=1$ or 2 (1:1 and 1:2) or bellows must be used (to have maximal magnification).

Of the four expressions of M_{transv}:

---

[340] In the ancient big cameras of the past or in the medium and high format camera with a big rear “box”, the minimum focusing distance was less than with compact cameras with the same *f*.

$$M_{transv}(f, s) = \frac{-f}{s-f}$$

$$M_{transv}(f', s') = \frac{-(s'-f')}{f'} = \frac{-s'}{f'} + 1$$

$$M_{transv}(f, Z) = \frac{-f}{Z}$$

$$M_{transv}(f, Z) = \frac{-Z'}{f'} = \frac{-Z'}{fn'/n}$$

we can use the ones function of (f, s) or (f, Z) to understand, for the two types of focusing (camera and eye), the change of M_{transv} when s (or Z) varies.

1. For the camera with f, f' fixed, we can use Fig.*APP-14* and one of the two expressions, $M_{transv}(f, Z)$ or $M_{transv}(f, s)$, with f constant. So:

$M_{transv,camera}(f_{camera}, s) = M_{transv}(s) = -\frac{f_{camera}}{s - f_{camera}}$ with f_{camera} constant. See also Fig.*APP-3*.

2. For the eye:

From: $\frac{n}{s} + \frac{n'}{s'} = \frac{n}{f}$, imposing $s' = f'_{Eye,\ max}$ = constant = 20mm[341], we obtain the variation of f_{Eye} with object distance (s):

$$f_{eye}(s) = \frac{f'_{eye,max}\, s}{\frac{n'}{n} s + f'_{eye,max}}$$

then[342]:

$$M_{transv,eye}(f_{eye}, s) = M_{transv}(s) = \frac{-f_{eye,max}}{s}$$ [343]

If we compare the two M_{transv} , with $f_{camera} = f_{Eye,\ max}$, $M_{transv,\ Eye} < M_{transv,\ camera}$[344] then also $|M_{long,\ Eye}| < |M_{long,\ camera}|$ with a better focusing and dimension adaptation. Autofocus of the eye makes what a good photographer normally makes (he shorts the focal length by a decreasing-zooming[345] when he approaches nearer objects).

---

[341] f' when the eye is uncommodated (objects at infinity).

[342] It seems that we can go near to $s = 0$, but this is only in the ideal (impossible) case of reducing f_{eye} to 0.

[343] This formula can be more simply derived from the (2ter) form of M_{transv} and from nodal point (in the eye nodal points and centre of pupils near coincide for small *f.o.v.*s).

[344] So, when we use a fixed focal length objective, the camera doesn't reproduce the delta magnification of the eye regarding fixed objects at different distances or, it is more evident in video-cameras, when we are moving respect to the scene. This is due to the different focusing mechanisms and can be proved by a practical experiment (for example, using a camera with a focal length equal to the relaxed eye (f = 15 mm), focusing far objects and then nearer objects before defocused: they increase dimensions more than eye; the same experiment is more significant with a video-camera). The graph of Fig.*APP-12* can be used, with $s' = f'_{max}$ = 20 mm and corrections in f and f'.

[345] The eye zoom is able do not move lenses (so not to elongate the entire eye dimensions). A practical experiment with camera can prove that it is possible to focus by using zoom, if zoom is made without moving

Fig. APP-18: Eye and camera magnifications using geometry

Fig. APP-19: Graphs for eye and camera magnifications

Fixed focus optical systems underlay the "artificial perspective rule" during normal shots (only one shot for the scene, with all objects in focus or out-of-focus). For video-camera, during dynamical shots the focal length must not change.

This difference is the reason why for using the trick of the puzzle composition in Par. 1.1.2.1.3 to obtain a high global *d.o.f.* to perfectly[346] respect the eye vision (so the *natural perspective rule*) (see Note 122) it would be necessary a focusing strategy using zoom like the eye carries out (*f.o.v.* of the eye is narrow, so they must use this trick, then have a changing focal length focusing). But changing focal length optical systems (as the eye) underlay the (artificial) perspective rule only in the theoretical case of changing focal length for each distance of the object space: this can be made by an eye (or camera trick) with infinitesimal *f.o.v.* and an infinitesimal zoom correction.

Perspective rule (*artificial* perspective) supposes also a perfect focusing → it is the perfect imaging with the correction of the transversal magnification to give images always on the same plane, as eye does.

Referring also to Fig. *APP-3*, this is the perspective rule:

H'. In any case the zoom change must be very small (see Note 127).

[346] See Note 68.

Fig. APP-20: $M_{transv,\, eye}$ and theoretical (artificial) perspective rule

But the zoom of a camera is a free degree of freedom[347], instead the zoom of the eye is a function of the object distance (s' is fixed). Starting from the transversal magnification:

$$M_{transv}(f', s') = \frac{-(s'-f')}{f'} = \frac{-s'}{f'} + 1$$

in the camera we have s' and f' free, so the zoom effect can be made by changing f with a zoom-multiplicative coefficient $K_{zoom}x$:

$$M_{transv}(f, s) = \frac{-K_{zoom} x f}{s - K_{zoom} x f}$$

We have, for long s respect to $K_{zoom} x f$, a near inverse-proportional relation for M_{transv}[348] [349] with $K_{zoom} x f$ as the coefficient.

APP.2.2 Combination of optical surfaces: global optical system parameters

First of all, the miracle of a projective transformation (so of imaging) is obtained also by a single optical spherical surface:

…...........

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[347] Limited by the limits of variations of s', as seen.

[348] For the eye the inverse proportionality vs. f (with s perfect but the zoom-multiplicative coefficient $K_{zoom}x$ is not free (it depends on s)).

[349] For example, a camera objective with a 15mm – 55 mm zoom, has a $K_{zoom} x = 3.67$. This means a change in magnification equal to up to 3.67 for objects >> 55 mm and less if s decreases. Eye has a $K_{zoom} x$ “zoom” equal to: 1.06, so very small (6% against 367%).

In case of refractive systems, to place objects and images in the same medium (normally air), we can use two or more surfaces. Besides they can be used to correct aberrations when non paraxial (real) bundles of rays are used.

Combination of two elements (surfaces or optical systems):

To proceed, it is necessary to calculate the global, equivalent *f*, *f* ', *H*, *H*' (n_L is the refraction index of the single lens or of the intermediate medium, *t* (also negative) is the separation between H_1' and H_2):

In case of simply two single surfaces[350] (→ thick lens), the dioptric power of the thick lens is:

$$P=\frac{n'}{f'}=\frac{n}{f}=P_1+P_2-t\frac{P_1P_2}{n_L}=\frac{n_L-n}{R_1}+\frac{n'-n_L}{R_2}+\frac{t}{n_L}\frac{(n_L-n)(n_L-n')}{R_1R_2} \quad [\text{Dioptres} = \text{m}^{-1}]$$

In the case of two systems or two thick lenses, the dioptric power of the complete system is:

$$P=\frac{n'}{f'}=\frac{n}{f}=P_1+P_2-t\frac{P_1P_2}{n_L}=\frac{n_L}{f_1'}+\frac{n'}{f_2}-t\frac{n'}{f_1'f_2'} \quad [\text{Dioptres} = \text{m}^{-1}]$$

$$h=t\frac{f}{f_2}=t\frac{P_2}{P}\frac{n}{n_L} \quad [\text{m}] \quad \text{is positive if inside}$$

$$h'=-t\frac{f'}{f_1'}=-t\frac{P_1}{P}\frac{n'}{n_L} \quad [\text{m}] \quad \text{is negative if inside}$$

with $t \geq 0$. For a thick lens, hiatus = $H' - H = t - h + h'$ [351]= $t[1-\frac{1}{P\,n_L}(n\,P_2+n'P_1)]$ (it can be also <0) (see cornea lens later).

Note that hiatus = 0 is a different concept respect to thin lens ($t = 0$): hiatus can be zero also with very thick lenses, as cylinders and spheres (see later the glass of water).

---

[350] In general for each surface, after the choice of a versus (left to right or vice-versa): $P_n = (n_{arrival} - n_{start})/R_n$.

[351] For a general system (see for example the eye in the following): $H' - H = hiatus = H' - H = t + hiatus_1 + hiatus_2 - h + h'$

Fig. APP-21: scheme and conventions for the combination of two elements

Important note:

All equations up to now (longitudinal, transversal, etc.) can be used with these global parameters with the condition that all surfaces are passed through by the rays that arrive to the eye/detector: see the following examples and see Fig. *APP-12*.

Fig. APP-22: special cases where not all surfaces are crossed

An instructive experiment: the glass of water

Instead of use a system composed by two or more lenses, it is simple to show the difference between a single surface and a complete system using a glass filled by water (n_{water} (20°C, λ = TOT) = 1.334). Putting a straw beyond the glass we have the complete system (the thick lens) ($f = f'$ = around $2R$, $H \equiv H'$ (→ hiatus = 0) in the centre of the glass), immersing the straw into the water we have only one surface and the straw is magnified because the f of the rear surface is greater (about $3R$) than the diameter of the glass, so this rear surface is used as a magnifier. Tilting the straw, we can observe the “prism effect”.

APP.2.2.1 Interesting particular cases

I. Plane-parallel sheet and telescopic systems

The plane-parallel sheet is a pure hiatus (its hiatus is equal to its depth), so it purely translates[352].

II. A (single) thick lens (and the “thin” lens)

A complex optical system is equivalent to a single thick lens: all effort to construct a complex optical system (as, for example, a photographic objective) seem to vanish, demonstrating the limit of the paraxial model (that doesn’t take into account, for example, of the aberrations).

Theoretical point:
As a particular case, if the thick of the lens is zero, the optical system reduces to the “thin” lens ideal model (from hiatus expression, it is zero).

III. Combinations of two thick lenses

…............

Example: the eye (cornea +crystalline lens)

The eye is an interesting example of the most general case (two (thick) lenses immersed in three different media) (in the camera usually there are more than two lenses, but they are in air)).

For the calculations, we can choose to couple the four interfaces (see Fig.*APP-23*) considering lenses the more dense couples, so the cornea and “eye lens”[353] [354]). The refraction index of aqueous and the vitreous humours, and above all of the tear film, can be estimated by the a solution of $H_2O + NaCl$ (TOT%) at the same λ and at $T = 37$°C. So the system can be be simply described by: “two low density lenses immersed in salt water, the first one with its first surface wet only by (a film of) salt water”.

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[352] See the piece of art by A. Paolini: *Refractio (Rifrazione)*, https://lartedelsorriso.eu/portfolio/opere-site-specific-e-a-tema/

[353] The crystalline lens is not the only lens in the eye (the cornea is the other one), but, due to its “typical” “lens shape” it is named, erroneously: “eye lens”.

[354] To be precise, a third lens is constituted by the tear film, and it can be added to the cascade after.

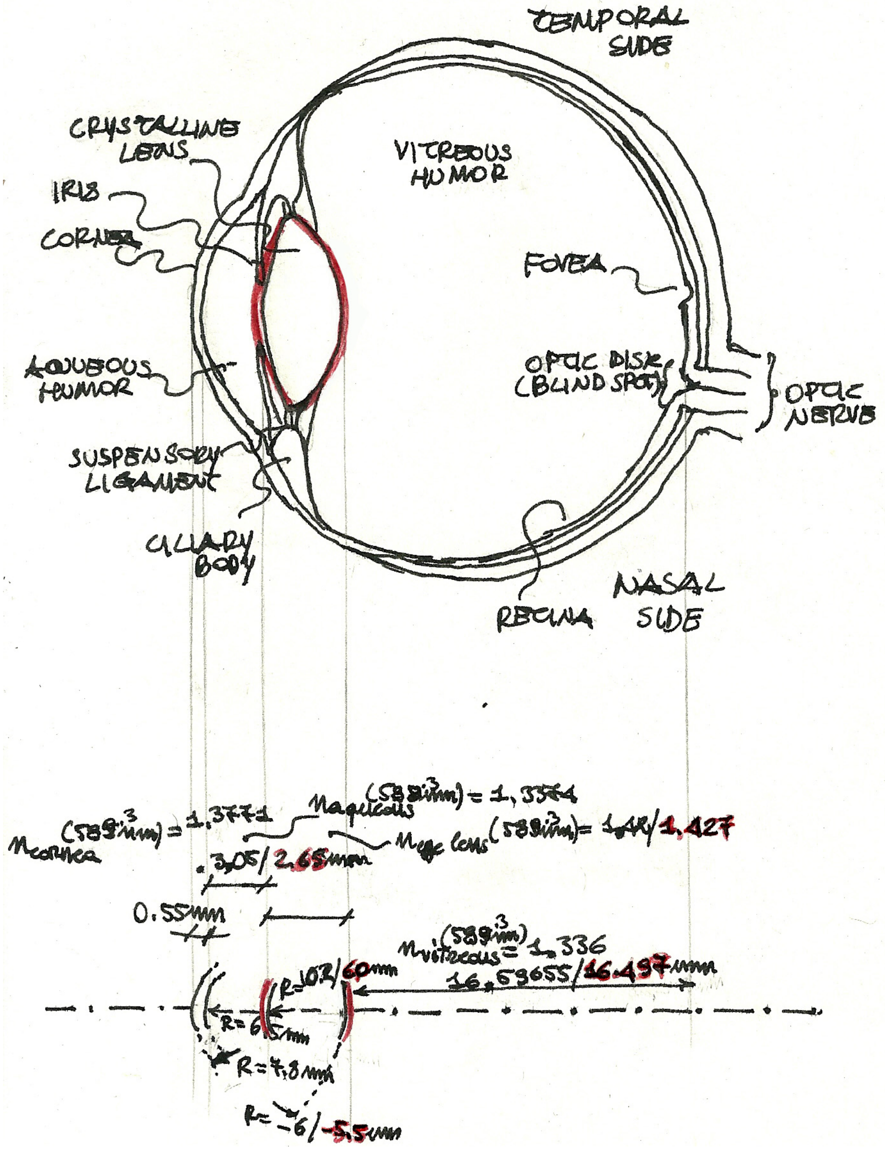

Fig. APP-23: Human eye: from the anatomy ((horizontal) section) to the paraxial model (LeGrand standardized paraxial model, here with refraction indexes calculated at $\lambda = 589.3$[355] nm). In black the uncommodated (relaxed ciliary muscle) case, in red the fully accomodated case. Front surface of the cornea can be determined by a corneal topography.

We calculate the focal lengths[356], the principal points and the nodal points for these two lenses, then for the total emmetropic eye. We will also calculate the pupils diameters and positions. Because of the focusing mechanism of the eye, these quantities (except for cornea) depend on the object distance: we will calculate them for the extreme cases: object at infinity (uncommodated eye) and object at minimum focusing distance (fully accomodated eye).

I. Uncommodated (relaxed) eye

From Fig. *APP-23* (we use the LeGrand (paraxial) model[357]):

1. For the *positive* meniscus cornea lens:

$P_{1cornea}$ ($\lambda = 589.3$ nm [358]) = +48.35 dioptres (the high refractive power of the anterior cornea surface is due to its high curvature and to the air[359] as the first medium) [in reality the first medium is the tear film, so another lens (the tear film) should be taken into consideration, see later].

$P_{2cornea}$ ($\lambda = 589.3$ nm) = -6.108 dioptres (also if the positive curvature of the posterior cornea surface is higher than the anterior one, its negative refractive power is much less, because the aqueous humor has a refractive index only slightly less than n_{cornea}, so the cornea lens is always a meniscus, but with a positive power (in the air a similar lens would have a negative power, it would be a "negative-meniscus")).

Finally, with $t_{cornea} = 0.55$ mm:

P_{cornea} ($\lambda = 589.3$ nm) = $P_{1cornea}$ ($\lambda = 589.3$ nm) + $P_{2cornea}$ ($\lambda = 589.3$ nm) – t_{cornea} $P_{1cornea}$ ($\lambda = 589.3$ nm) $P_{2cornea}$ ($\lambda = 589.3$ nm) / n_{cornea} ($\lambda = 589.3$ nm) = +42.3699 dioptres (the cornea has a power greater than the crystalline lens, see later),

so:

f_{cornea} ($\lambda = 589.3$ nm) = + 23.6 mm

---

[355] One particular wavelength (here yellow) is chosen in coherence to the ideal model of perfect imaging (paraxial model). In reality dispersion (chromatic aberration) is present. This wavelength should be the weighted value according to dispersion (but different media have in general different dispersions).

[356] In paraxial optics astigmatism should be considered: see A.1.5.1.2 for first surface corneal astigmatism, but in the model the eye is an emmetropic one (without astigmatism).

[357] In the (full) Gullstrand model the crystalline lens is contained, as in reality, in a capsule.

[358] $T = 37$°C is subtended.

[359] When we open our eyes under water, this high power is losen ($P_{1cornea}$ falls down to around 5.5 Dioptres, with $P_{eye,\ uncommodated}$ that looses around 42 Dioptres: short-sighted persons are a little advantaged under water (they have less hyperopia under water)).

f'_{cornea} (λ = 589.3 nm) = + 31.5648 mm

h_{cornea} (λ = 589.3 nm) = - 0.05759109999 mm
h'_{cornea} (λ =589.3 nm) = - 0.60967933113 mm
$hiatus_{cornea}$ (λ =589.3 nm) = t_{cornea} - h_{cornea}+ h'_{cornea} = - 0.00208823114 mm (negative)

Seeing the position of the iris[360], only the cornea lens determines the diameter and position of the entrance pupil (that are independent from accomodation): measuring s_{iris} from H'_{cornea} (because the "half-space" is here the object space):
s_{iris} = 0.60967933113 mm + 3.05 mm (with the iris at 3.6 mm from the first cornea vertex, so in contact with the crystalline vertex[361]) = 3.65968 mm

→ the virtual image is at: $s'_{iris} = 1/(-(n_{aqueous}/s_{iris}) + P_{cornea}))$ = - 3.095286 mm from H_{cornea}, so at 3.038 mm behind the first cornea vertex
→ M_{transv}(iris, from entrance) = $\frac{-f'_{cornea}}{s_{iris}-f'_{cornea}}$; using f'_{cornea} instead of f_{cornea} , = +1.13115, so about 13.1%

To resume: the iris is "imaged" slightly nearer and is a little increased.

2. For the double-convex positive (here uncommodated) crystalline lens:

Whereas with accomodation power and principal plane positions of the cornea do not change, for eye lens, refractive power changes because surface curvatures and width change.

$P_{1\ eye\ lens,\ uncommodated}$ (λ = 589.3 nm) = +8.1 dioptres (refractive power of the first surface of the eye lens is not so high despite its high curvature, because of the aqueous humour)
$P_{2\ eye\ lens,\ uncommodated}$ (λ = 589.3 nm) = +14 dioptres (also if the curvature of the posterior surface of the eye lens is high, its positive refractive power is not so high, because the vitreous humour has a refractive index not so less than $n_{eye\ lenss}$).
Then: $P_{eye\ lens,\ uncommodated}$ (λ = 589.3 nm) = $P_{1eye\ lens,\ uncommodated}$ (λ = 589.3 nm) + $P_{2eye\ lens,\ uncommodated}$ (λ = 589.3 nm) - $t_{eye\ lens,\ uncommodated}$ $P_{1\ eye\ lens,\ uncommodated}$ (λ = 589.3 nm) $P_{2\ eye\ lens,\ uncommodated}$ (λ = 589.3 nm) / $n_{eye\ lens\ (uncommodated)}$ (λ = 589.3 nm) = +21.78056 dioptres

$f_{eye\ lens,\ uncommodated}$ (λ = 589.3 nm) = +61.40338 mm
$f'_{eye\ lens,\ uncommodated}$ (λ = 589.3 nm) = +61.3391 mm

(if it would be in air, its focal lengths would be around 46 mm).

Also positions of principal planes change with accommodation.

For an uncommodated eye, we have:

---

[360] variable in diameter

[361] This is not coherent with the 0.4 mm advancement of the first vertex of the crystalline lens during accomodation. So, in reality, M_{transv} (iris) is less than +13%.

$h_{eye\ lens,\ uncommodated}$ (λ = 589.3 nm) = +2.42175 mm
$h'_{eye\ lens,\ uncommodated}$ (λ = 589.3 nm) = - 1.39957 mm
$hiatus_{eye\ lens,\ uncommodated}$ (λ = 589.3 nm) = $t_{eye\ lens,\ uncommodated}$ $-h_{eye\ lens,\ uncommodated}$ $+\ h'_{eye\ lens,\ uncommodated}$ = +0.17868 mm

Now, we can calculate the parameters for the composition of these two optical subsystems (thick lenses), then the parameters of the complete eye.

From the previous calculations, the distance between the cornea second principal plane and the crystalline first principal plane is:

$t_{H'cornea\text{-}Heye\ lens,uncommodated} = t_{aqueous,\ uncommodated} - h'_{cornea} + h_{eye\ lens,\ uncommodated}$ = 3.05 mm + 0.60967933113 mm + 2.42175 mm = +6.0814 mm

$P_{eye,\ uncommodated} = P_{cornea} + P_{eye\ lens,\ uncommodated} - t_{H'cornea\text{-}Heye\ lens,\ uncommodated}\ P_{cornea}\ P_{eye\ lens,\ uncommodated} / n_{aqueous}$ = +59.945 D

so:

$f_{Eye,\ uncommodated}$ = +16.68196 mm (so the eye is as a camera with this *f*, so a compact camera; note that *f.o.v.* of the eye is much greater than compact camera because the retina is curved, so it can be compacted in the round shape of the eye)
$f'_{Eye,\ uncommodated}$ = +22.2871 mm

and:

$h_{Eye,\ uncommodated}$ = +1.6533551916 mm, so the distance of the first principal plane from the first vertex of the cornea is +1.595764 mm[362]
$h'_{Eye,\ uncommodated}$ = -4.2939 mm, then we can calculate the standard emmetropic eye length from the fact that $F'_{Eye,\ uncommodated}$ is on the retina:

24.19655 mm

$hiatus_{Eye,\ uncommodated}$ (λ = 589.3 nm) = $H'_{Eye,\ uncommodated}$ - $H_{Eye,\ uncommodated}$ = $t_{H'cornea\text{-}Heye\ lens}$ + $hiatus_{cornea}$ + $hiatus_{eye\ lens,\ uncommodated}$ $- h_{Eye,\ uncommodated}$ $+ h'_{Eye\ lens,\ uncommodated}$ = 0.3107 mm (so $H'_{Eye,\ uncommodated}$ is 1.90653 mm inside the first cornea vertex).

To resume, the uncommodated eye has *f* = around +17 mm, with hiatus around zero and principal planes inside the vertex of around 1.6 mm.

---

[362] So F_{Eye} is 15.086 mm in front of the cornea first vertex: this is the Knapp distance, the ideal position of the *H'* of the ophthalmic lens in case of a purely axial refractive defect (see: A. Paolini, *L'incidente stradale*, Mathforlife.net Project (https://archive.org/details/IndaginiDiUnRicercatoreprimoEpisodio).

With these data, we can derive the nodal point $N_{Eye,\ uncommodated}$: $|N_{Eye,\ uncommodated} - H_{Eye,\ uncommodated}| = f_{Eye,\ uncommodated} - f'_{Eye,\ uncommodated}$ = 16.68196 mm - 22.2871 mm = - 5.60514 mm (so it is 7.2 mm inside the first cornea vertex), and $N'_{Eye,\ uncommodated}$: $|N'_{Eye,\ uncommodated} - H_{Eye,\ uncommodated}| = |N_{Eye,\ uncommodated} - H_{Eye,\ uncommodated}| - hiatus_{Eye,\ uncommodated}$ (λ = 589.3 nm)= - 5.91584 mm (so it is 7.5107 mm inside the first cornea vertex).

Position and dimension of the exit pupil, E' [363], can be calculated seeing the iris from the right, so using the iris position respect to $H_{eye\ lens,\ uncommodated}$ (s_{iris} [364]= +2.42175 mm) → s'_{iris} [365] = - 2.518546 mm (so exit pupil (virtual image) is 2.60043 mm - 2.518546 mm = 0.08188 mm inside the first eye lens vertex), or calculating the image of the calculated entrance pupil formed by the entire, uncommodated, eye: s_E = -1.442236 mm → $s'_{E,\ uncommodated}$= +1.7735 mm (considering that $H'_{Eye,\ uncommodated}$ is 1.90653 mm inside the anterior cornea vertex, the exit pupil of the uncommodated eye is 3.68 mm from it). Using $M_{transv,\ uncommodated}$(iris, from exit) = +1.041, and considering the M_{transv}(iris, from entrance) already calculated:

$P_{M,Eye,\ uncommodated}$ = 1.041 / 1.13115 = 0.92

so in the uncommodated eye the aperture stop (iris) is putted in a slight "asymmetrical way vs. magnification"[366]. Note that, from (2bis), knowing M_{transv} we can calculate s_E and s_E', but because normally P_M is only positive[367], we can calculate the positions of the pupils but with an indetermination due to the sign of P_M:

For the entrance pupil: $s_E = f\dfrac{(P_M - 1)}{P_M}$, and for the exit pupil: $s'_E = (1 - P_M) f'$

See Fig. *APP. 24*.

... with the tear film (IN THE FOLLOWING there is an error: tear film I reality changes the curvature, see the glasses example, so the formula have To Be Corrected)

We will approximate that the tear film increases only the thickness of the cornea (so we suppose that $n_{tear\ film}$ (λ = 589.3 nm) = n_{cornea} (λ = 589.3 nm) and that $R_{1\ tear\ film} = R_{2\ tear\ film} = R_{1\ cornea}$[368])]. In reality, the cornea is not in air, but there is the tear film. So, we can calculate again the principal planes and the focal length of the eye when the first mean is now the tear medium and not the air (n=1.3386 instead of n=1; tear has about the same index of refraction of the vitreous humour), so the only quantity that changes is $t_{cornea} \rightarrow t_{cornea,\ tear}$

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[363] For the transitive property of conjugate points (the entrance pupil is the image of the aperture stop seeing from the left and the exit pupil is the image of the iris seeing from the right), the exit pupil is the image (due to the entire system) of the entrance pupil; for this reason we name the exit pupil E'.

[364] We name this distance s but we could use also s' (longitudinal euqation is symmetric respect $s \leftrightarrow s'$)

[365] This is the symmetric of Note 214.

[366] See Note 102.

[367] See Note 100.

[368] $R2_{tear\ film} = R1_{cornea}$ is true: the astigmatism of the first corneal surface can be eliminated by the tear film (see rigid contact lens properties). $n_{tear\ film}$ (λ =???) = n_{cornea} (λ =???) is also nearly true: 1.3386 vs. 1.3771.

[No: see water glasses curvature]

For an unacommodated eye:

For the cornea lens, now focal length and cardinal points positions change:

$P_{1cornea}$ = ??? dioptres (as without tear film)
$P_{2cornea}$ = ??? dioptres (as without tear film).
Then: $P_{cornea,\ tear} = P_{1cornea} + P_{2cornea} - t_{cornea,\ tear}\, P_{1cornea}\, P_{2cornea} / n_{cornea} =$
??? dioptres (now the refractive power of the cornea is...),
so:

$f_{cornea,tear}$ = +??.... mm
$f'_{cornea,tear}$ = ... mm

$h_{cornea,tear}$ = ??? mm
$h'_{cornea,tear}$ = ??? mm
$hiatus_{cornea,tear} = t_{cornea,\ tear}$ -....$h_{cornea,tear}$...$h'_{cornea,tear}$ = ????

……………..

For the double-convex positive (uncommodated) eye lens, the parameters are the same as calculated before, so:

….........

Adding the tear film[369] *(uncommodated eye with tear film)*

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[369] Note on the beautiful of nature: cornea and other eye surfaces constitute an undesired katoptric system due to their Fresnel reflections (see Note 317). The first surface of the system (in reality tear film and not cornea) is the major contributor, above all on the dark circle of the pupil, where the contrast is higher (and a little also on the iris, if it is dark). This reflection makes vivid the pupil that otherwise would be similar to a “black body” (for the possibility to see inside himself, see: A. Paolini, www.lartedelsorriso.eu). If we see, at a short distance, another person, or ourselves, straight into the eyes, we can see the “pupil”, so our face reflected. Indeed, the first surface of cornea has such a high curvature ($r_{1\ cornea}$ = 7.8 mm) that its convex reflecting surface creates such a very minimized virtual images that we have to stay at as short distance as is possible to see our own face reflected, so the best is stay at 25 cm (if we use a planar mirror, at 25/2 = 12.5 cm from the mirror) (at higher distances, the *f.o.v.* of the convex mirror captures more than our face and our face is too small to be recognised). If, instead, we use a non-planar mirror (a mirror with a magnification factor different by 1, variable with the distance from the mirror and with F), there are two possibilities: a convex mirror, but it always mignifies, so the reflection on the pupil is always smaller than with nuke eye at 25 cm (at 25/2 = 12.5 cm using a planar mirror); a concave mirror, different if we used it as a magnifier (with the eye positioned after the focus): if focal length is more than 25 cm, I can simply approach eye; but if less than 25 cm, the angular magnification respect to the nuke eye vision at 25cm is greater: in this case I see my eye bigger than I can see it with nuke eye, and onto the pupil I see a smaller part of my face, until to arrive to see only my cornea (see: A. Paolini, *cit.*).
Besides this, if we use (at least) one mirror to see “ourselves” (more than only our pupil) on “our” pupil, there are always (also if the concave mirror “compensates” the convexity of the cornea, because of the positional

For the complete uncommodated eye with tear film: ... **Warning: the tear film changes the curvatures (see glass example)**

From the previous calculations, the distance between the principal planes is:

To be corrected:
$t_{H'\text{cornea-}H\text{eye lens,tear}}$ = 3.05 mm -3.4090379764 mm + 2.42175 mm = +2.062712036mm

$P_{\text{Eye,tear}} = P_{\text{cornea,tear}} + P_{\text{eye lens}} - t_{H'\text{cornea-}H\text{eye lens,tear}}\, P_{\text{cornea,tear}}\, P_{\text{eye lens}} / n_{\text{aqueous humor}} = +$ 20.9826864601 D

so:

$f_{\text{Eye,tear}}$ = +???63.6715418946 mm
$f'_{\text{Eye,tear}}$ = +..... mm

and, using the parameters calculated before:

$h_{\text{Eye,tear}}$ = +2.14114742174 mm so the distance of the first principal plane from the first vertex of the cornea is +3.94736157722 +2.14114742174 mm = +6.08850899896 mm
$h'_{\text{Eye,tear}}$ = ... mm
To be corrected

Observation: when the tear film is "thicker" (when we cry), the optical power is lower (see formula [APP. X] and discussion), so myopic eye sees better (emmetropic eye is able to correct for this little hyperopic effect).
When we do not cry, the mean ± SD pre-corneal tear film thickness is 3.3 ± 1.5 µm (range, 0–6.9), so -0.0022326D is the (mean) contribution of the pre-corneal tear film. If we increase the tear film with moistening this contribution can increase, but the real improvement is due to the decrease of the first corneal astigmatism (see Note 58).

II. Fully accommodated eye

Whereas with accommodation cornea optical power and cornea principal plane positions do not change, for crystalline lens optical refractive power and positions of principal planes change with accommodation because surface curvatures change (especially front surface: of the 0.5 mm total central delta-thickness of the crystalline lens, 0.4 mm are due to the first surface curvature increase, see *Fig.APP*-23).

---

offset) infinite pupils with images than chase each other inside them...

2bis. For the double-convex positive (here fully accommodated) crystalline lens:

$P_{1\ \text{eye lens, fully accommodated}}$ (λ = 589.3 nm) = +14.93 dioptres (with an increase of 6.83 dioptres respect to the uncommodated case)
$P_{2\text{cornea, fully accommodated}}$ (λ = 589.3 nm) = +16.545 dioptres (+2.545 dioptres respect to the uncommodated case).
Then: $P_{\text{eye lens, fully accommodated}}$ (λ = 589.3 nm) = $P_{1\text{eye lens, fully accommodated}}$ (λ = 589.3 nm) + $P_{2\text{eye lens, fully accommodated}}$ (λ = 589.3 nm) - $t_{\text{eye lens, fully accommodated}}\, P_{1\ \text{eye lens, fully accommodated}}$ (λ = 589.3 nm) $P_{2\ \text{eye lens, fully accommodated}}$ (λ = 589.3 nm) / $n_{\text{eye lens (fully accommodated)}}$ (λ = 589.3 nm) = +30.696 dioptres (+8.915 dioptres respect to the uncommodated case).

$f_{\text{eye lens, fully accommodated}}$ (λ = 589.3 nm) = +43.569 mm
$f'_{\text{eye lens, fullt accommodated}}$ (λ = 589.3 nm) = +43.5235 mm

Also positions of its principal planes change with accommodation:

$h_{\text{eye lens, fully accommodated}}$ (λ = 589.3 nm) = +2.27318 mm
$h'_{\text{eye lens, fully accommodated}}$ (λ = 589.3 nm) = -2.04915 mm
$hiatus_{\text{eye lens, fully accommodated}}$ (λ = 589.3 nm) = $t_{\text{eye lens, fully accommodated}} - h_{\text{eye lens, fully accommodated}} + h'_{\text{eye lens, fully accommodated}}$ = +0.177668 mm

Now, we can calculate the parameters for the composition of these two optical subsystems (two thick lenses), then the parameters of the fully accommodated complete eye, without tear film.

From the previous calculations, the distance between the cornea second principal plane and the crystalline first principal plane is:

$t_{H'\text{cornea-}H\text{eye lens, fully accommodated}} = t_{\text{aqueous, fully accommodated}} - h'_{\text{cornea}} + h_{\text{eye lens, fully accommodated}}$ = 2.65 mm + 0.609679 mm +2.27318 mm = +5.532859 mm

so:

$P_{\text{eye, fully accommodated}} = P_{\text{cornea}} + P_{\text{eye lens, fully accommodated}} - t_{H'\text{cornea-}H\text{eye lens, fully accommodated}}\, P_{\text{cornea}}\, P_{\text{eye lens, fully accommodated}} / n_{\text{aqueous}}$ = +67.685 D (against +59.945 D of the uncommodated case)

and:

$f_{\text{Eye, fully accommodated}}$ = +14.7742 mm (against +16.68196 mm of the uncommodated case)
$f'_{\text{Eye, fully accommodated}}$ = +19.738 mm (against +22.2871 mm of the uncommodated case)

$h_{\text{Eye, fully accommodated}}$ = +1.87619 mm, so it moves inside of 0.22 mm respect to the uncommodated case (the distance of the first principal plane from the first vertex of the cornea goes to +1.8186 mm[370])
$h'_{\text{Eye lens, fully accommodated}}$ = -3.45987, so 0.83 mm more distant to the first cornea vertex respect to the uncommodated eye. Considering that the standard emmetropic eye length is always the same (24.19655 mm), we can calculate where $F'_{\text{Eye, fully accommodated}}$ is before the retina:

-2.26758 mm

and calculate which is the distance of the nearer focused point of the LeGrand fully accomodated model (its $s'_{\text{focused, near point,}}$ is 22.00558 mm):

$s_{\text{focused, near point,}}$ = 143.4 mm, so 141.58 mm from the first vertex of the cornea.

$hiatus_{\text{Eye, fully accommodated}}$ (λ = 589.3 nm) = $H'_{\text{Eye, fully accommodated}} - H_{\text{Eye, fully accommodated}} = t_{H'\text{cornea-Heye lens, fully accomodated}} + hiatus_{\text{cornea}} + hiatus_{\text{eye lens, fully accommodated}} - h_{\text{Eye, fully accommodated}} + h'_{\text{Eye lens, fully accommodated}}$ = 0.372378 mm (so $H'_{\text{Eye, fully accommodated}}$ is 2.19097 mm inside the first cornea vertex).

To resume, the fully accomodated eye has f around +15 mm, with a hiatus around zero and principal planes inside the vertex of around 2 mm.

With these data, we can derive the nodal point $N_{\text{Eye, fully accommodated}}$: $|N_{\text{Eye, fully accommodated}} - H_{\text{Eye, fully accommodated}}| = f_{\text{Eye, fully accommodated}} - f'_{\text{Eye, fully accommodated}}$ = 14.7742 mm - 19.738 mm = - 4.9638 mm (so it is 6.78 mm inside the first cornea vertex), and $N'_{\text{Eye, fully accommodated}}$: $|N'_{\text{Eye, fully accommodated}} - H_{\text{Eye, fully accommodated}}| = |N_{\text{Eye, fully accommodated}} - H_{\text{Eye, fully accommodated}}| - hiatus_{\text{Eye, fully accommodated}}$ (λ = 589.3 nm)= - 5.3362 mm (so it is 7.152378 mm inside the first cornea vertex).

For position and dimension of the exit pupil, E', we use the method that calculates it seeing the iris from the right, so using the iris position respect to $H_{\text{eye lens, fully accommodated}}$ (s_{iris}[371] = - 1.87318 mm) → s'_{iris}[372] = - 1.95528 mm (so exit pupil (virtual image) is 2.45085 mm - 1.95528 mm = 0.49557 mm inside the first eye lens vertex). Using $M_{\text{transv, fully accommodated}}$(iris, from exit) = +0.9588, and considering the M_{transv}(iris, from entrance) already calculated:

$P_{M,\text{Eye, fully accommodated}}$ = 0.9588 / 1.13115 = 0.8476

so also in the fully accommodated eye the aperture stop (iris) is putted in an “asymmetrical way vs. magnification”[373].

---

[370] So F_{Eye} decreases to from 15.086 mm to 12.955 mm in front of the cornea first vertex: the Knapp distance, the ideal position of the H' of the ophthalmic lens in case of a purely axial refractive defect, decreases if the object approaches (see: A. Paolini, *L’incidente stradale*, Mathforlife.net Project (https://archive.org/details/IndaginiDiUnRicercatoreprimoEpisodio).

[371] We name this distance s but we could use also s' (longitudinal equation is symmetric respect $s \leftrightarrow s'$). Iris position is supposed do not change with accomodation (see Note 230).

[372] This is the symmetry of Note 214.

[373] See Note 102.

In the fully accomodated eye *E'* diminishes of 8% and moves away of 1.225mm from *H'*, respect to the uncommodated case, see Fig. *APP. 24*.

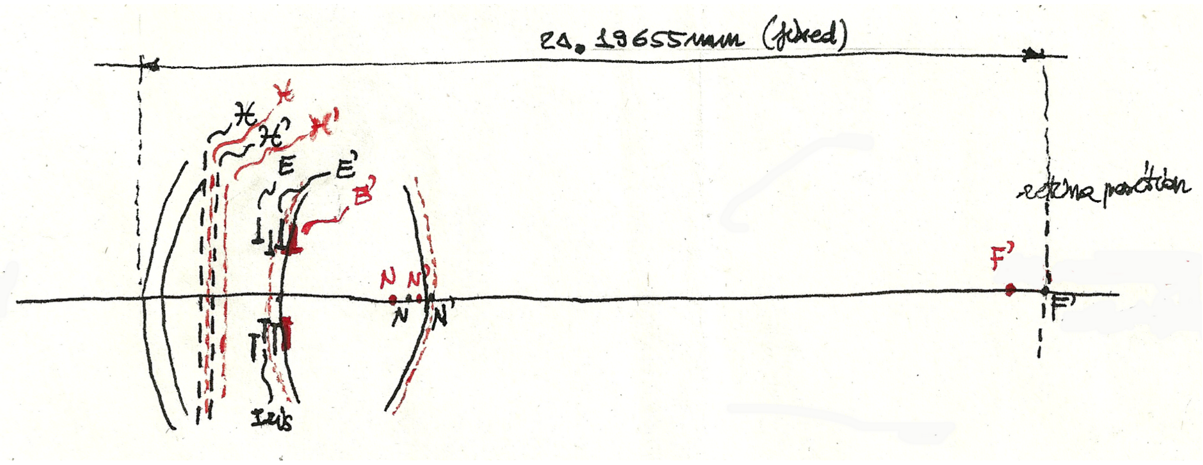

Fig. APP-24: The LeGrand standardized paraxial model with cardinal points, without the tear film, here with refraction indexes calculated at λ = 589.3[374] *nm. Also pupils have been added. In black the uncommodated (relaxed ciliary muscle) case, in* ***red*** *the fully accomodated case.*

Adding the tear film (fully accommodated eye with tear film)

[…]

- The corrected myopic and hyperopic eyes

It is simple to add to the eye the ophthalmic lens. Mind the “possible” change in magnification (see Knapp[375] distance).

APP.3 Wave optics, diffraction limited systems

The functions that characterize a linear[376], spatial-invariant reproduction system in space domain, in particular a circular symmetric one, are: the *P.S.F.* or the *L.S.F.* or its F^{-1} = *MTF*.

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[374] One particular wavelength (here yellow) is chosen in coherence to the ideal model of perfect imaging (paraxial model). In reality dispersion (chromatic aberration) is present. This wavelength should be the weighted value according to dispersion (but different media have in general different dispersions).

[375] See: A. Paolini, *L'incidente stradale*, (https://archive.org/details/IndaginiDiUnRicercatoreprimoEpisodio), cited.

[376] Optical systems are nearly spatially linear (with a constant time-frequency spectrum) and this is proved by the fact that geometry can be applied in optics (see geometrical optics) by dividing the object in points and each one can be treated separately from the others. For this reason normalized *MTF* (for a certain time-spectrum source) can be used for optics.

APP.3.1 Optical performance in wave optics and the theory of information

The information quantity about the scene can be divided into: time information and spatial information[377]. Here we will limit our analysis to the spatial component, in particular to the scene information diminution due to the unavoidable detector (eye) or to the reproduction system (camera or camera+eye).

The spatial information is brought by all parameters associated to the image, so sharpness and energetic parameters. Sharpness degradation due to the optics, influenced also by aberrations, defocus, distortion, etc., can be analysed by wave optics using, in incoherent regime, the Point Spread Function[378], the Line Spread Function[379] and, because with a high extent optical systems are linear, efficaciously it can be expressed, in the domain of spatial frequencies[380] [381] [382], by the module of its Fourier transform, $MTF_{eye,optics}$ or $MTF_{camera,optics}$ (chromatic aberration can be taken into consideration giving the *MTF* at various wavelengths, or practically giving the *MTF* calculated for the same source spectrum (for example, white light with a certain spectrum)). The contribution of the detector must be added to obtain the complete *MTF*, then the total lost of "sharpness" information.

Quantities related to power (aperture efficiency, relative illuminance, or other quality parameters (colours reproduction, flare) can be calculated more precisely by wave optics.

---

[377] The two are connected in quantic physics, but we can hypothesise that they are independent, or also not because we analyse the spatial component only. Both of them have an influence on the speed resolution, so in the Hamiltonian/Lagrangian coordinates.

[378] An axial symmetric system is axial symmetric if the point is on axis (see cat-eye shapes in the practical following experiments if the punctiform light is not on axis). This is more evident for not-diffraction limited systems, where refraction contributes to the off-axis worsening. The corresponding *MTF* depends on the out-of axis amount [it is not isotropic (it does not depend on the direction of the sine grating)].

[379] It is linked to *P.S.F.* (it is the convolution of it with a line).

[380] The loose of information is more evident when we stay in the space domain and not in the spatial frequencies domain: the point spead function shows that a point object is always spread into a **larger** image, so if we use the resolving power of the system instead of the (equivalent aspect of) sharpness, the theoretically infinite value is decreased to a really finite value, then the information quantity of the scene is always degraded (the spatial resolution decreases).

[381] The *MTF* in spatial frequencies is the correspective of the transfer function (frequency response) in time frequencies domain used in electronics and in physics, for example in acoustics/sound reproduction.

[382] It is a function to measure the fidelity of reproduction.

Fig. APP-25: If the partition becomes denser, information entropy increases. The quantity of information (entropy of information) is conventionally measured in bits (by base 2 logarithm)

If, finally, the reproduction is seen by the eye, there is a total information quantity limit that is: $MTF_{eye,total}$ x Eye field of vision. See C1.

APP.3.1.1 The MTF of an optical system

MTF is a good global function to judge (only spatially[383]) the optical performance of a static[384] imaging systems (on focus or defocused), also if it is not sufficient alone to characterize it. Its degradation is influenced by "macro" geometrical imperfections (the optical aberrations) (that depends on *f#*) and inherent physical limitations, due to diffraction (for distant[385] objects or at shorter distances if there is an optical system, it depends exclusively by D/λ[386]).

MTF is a function of two spatial frequencies, but for a centred revolution system (around one axis)[387] the function is centred symmetric, so only one sagittal section can be taken and the *MTF* is function of only one spatial frequency. Spatial frequency is here expressed as an angular[388] spatial frequency [cycles/°]. Spatial frequency expressed in mm/° is convenient only for the *MTF* of the detector (for the eye: MTF_{retina}; for the camera: MTF_{sensor} x MTF_{paper} or x MTF_{screen}) and, for $MTF_{eye,optics}$ or $MTF_{camera,optics}$, when they must be multiplied by $MTF_{detector}$ to obtain the global *MTF*.

APP.3.1.1.1 MTF of an aberration free optical system (on focus and out of focus diffraction limited optical system)

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[383] The eye, as the camera, is also a temporal system.

[384] Stationary.

[385] Not infinite.

[386] It depends as the wave "sees the diameter". Because of the presence (refraction capacity) of the optical system (see Fraunhofer and Fresnel diffraction) *MTF* doesn't depend by *f* and by object distance .

[387] So a system without astigmatism.

[388] Both for total eye and total camera, because here it is supposed that photographies are seen respecting the *P.o.V.* position (respecting the perspective).

There is a limitation[389] to optical information conservation due to physics: a finite aperture (with or without optics[390]) corrupts the optical information content of the object[391]. Optics can be useful to increase the *f#*, so to compact a system that was larger, but introduces aberrations. This discussion is true for an on-focus system, so with the goal to conserve as mush as possible the object (spatial) information content. Indeed, in out-of-focus systems (camera or eye), or in refractive defective systems (myopic or hyperopic eye), there is the instinct to decrease the pupil diameter, because the Fresnel point spread function (the "blur circle" in geometrical optics) depends on *f#* and increases in width when *f#* decreases.

[...]

Fig. APP-26: Fraunhofer (focused image) and Fresnel (out-of-focus image) MTFs and P.S.F.s

Practical experiments:

The Fresnel[392] point spread function (inverse Fourier transform of the module of the "Fresnel" *MTF*) can be personally experimented by nuke eye if you have a diffraction defect (myopia or hyperopia), in the following manner:

- Go in a dark ambient and adapt to it staying some time (your pupil diameter enlarges, so the *f#* decreases, and the point spread function will enlarge and will be more intense)
- Use a point light source (as a Christmas tree lamp or another punctiform lamp): place it far if you are near-sighted or near if you are far-sighted: you will see, in a good approximation, the Fresnel point spread function[393] of your eye[394]. If the light is intense[395], and

---

[389] Diffraction limited optical systems are only a theoretical hypothesis because before there is the "geometrical" (but really physical) limit of aberrations.

[390] See the cameras without optics: the *camera obscura* that must be big as a room to give good sharpness, or its compact version, the pinhole camera, that due to diffraction gives poor sharp images (and too dark).

[391] Field means that all is "connected/related bewteen" in time-space, in particular when the field propagates (in the form of waves): because the wavefront is "coherently connected by phase", also optical amplitude is corrupted by the edges of the hole, so also using simply incoherent detectors we undergo a decrease of detected information.

[392] The Fresnel point spread function is more spread than the Fraunhofer point spread function and is the only one visible with eye or camera: the Fraunhofer point spread function can not be visible (it is the "base" of the sharp vision).

[393] According to your entrance pupil diameter and your refractive "defect" amount.

[394] Complete of brain.

[395] But do not fix too long the source, if not your pupil will decrease diameter and the effect will be less visible. Also do not half-close your eyelids (do not squint) as normally far-sighted and near-sighted do to decrease your (entrance) pupil, so to sharp images.

the ambient very dark[396], you can see[397] not only the geometrical single "blur circle[398]", but also one or more additional[399] circles of the wave Fresnel point spread function.

If your "defect" is high, the effect will be more evident (beautiful) (a storm of "blur circles[224]" in the case of the Christmas tree).

The Fresnel point spread function can also be studied by a camera, in a similar manner:

- Go in a dark[400] ambient and take, with a tripod, a photo of a point light source[401] (as a Christmas tree lamp or another punctiform lamp (without reflector or collimator)), with manual focus setted on a shorter distance than the right one[402] (→ as near-sighted) (the far-sighted simulation is, as in the eye, less effective because the *d.o.f.* is not symmetrical), changing if you want the diaphragm aperture (with larger apertures the point spread function will enlarge[403]). [Judge the effect directly in the viewfinder or on the LCD or later on the photo]. To amplify the effect (to decrease the *d.o.f.*), use long focal lengths.
- On the photography, you will see, in a good approximation, the Fresnel point spread function[404] of the complete camera[405]. If the sensor[406] has a good relative and absolute sensitivity and a high dynamics (in the best case, noise is the limit), and the ambient is really dark, you can see in the photo or on the LCD not only the geometrical single "blur circle", but also one or more additional circles of the wave Fresnel point spread function. This is possible if considering dynamics and relative intensity changes, a Fresnel pattern can be seen by an emmetropic nuke eye.

By the camera is possible to simulate the refractive eye defects[407] and also the diffractive eye defects, stay that no astigmatism is present.

APP.3.1.2.1 Influence of the five aberrations on the MTF

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[396] Do not perturb the relative (luminance response) and absolute sensitivity of the (total) eye.

[397] $MTF_{eye,total}$ can be the limit or, deeper, the noise of the retina+brain.

[398] It is a circle if the eye is not astigmatic, otherwise you will see a "blur ellipse", different from left or right eye if their astigmatism amount is different.

[399] If the eye sensitivity is enough.

[400] With the camera, the darkness is not used to dilate the pupil, but do not have other lights that over-impose on the Fresnel pattern.

[401] With camera you can also study the influence of the direction: in-axis or out-of-axis.

[402] With the objective all-out.

[403] At maximum aperture the diaphragm is circular, at lower apertures it is a polygon.

[404] According to the *d.o.f.*, so: the *f#*, the focal length and the object distance.

[405] Plus media to show it.

[406] To see directly these fine effects by the eye in the viewfinder or LCD, see experiment by nuke eye (light must be intense and not fixed in permanence).

[407] See the piece of art: A. Paolini, *Il mio Natale* [*My Christmas*], not yet published.

APP. 4 Elements of perspective (projection and topography)

The perspective, using projection techniques, is useful only to give to the spectator the illusion to see the scene from a certain $P.o.V.|_{3D\ scene}$. The scene can exist (see the following picture on the life painting/design, where photography+eventual-post-processing is much more faster than painting/drawing) or can be invented by the designer (a 3D "reality"), for example by orthogonal projections/views (see the following picture): in this case photography can not intervene, and the invention can be projected by perspective using one of the five methods of construction.

Fig. APP-27: Picture with a man that sees through an ocular and a squared glass a 3D scene that exists

Fig. APP-28: This method, described here in a picture of .... (15...) but known from Leon Battista Alberti (14...) is for life painting/design and can be approximated in an instant by photography.

Fig. APP-29: The five methods to project an invented 3D scene according to a certain $P.o.V.|_{3D\ scene}$.

APP.5 Elements of the theory of information

Information coming from object half-space (position, frequency, luminance[408]) is proportional to the partitions (*measures*) that we make of these quantities. To detect/see/measure object space by an optical system in front of the detector, we have to add optical system information conservation capability to the detector one. In the ideal case of linear systems[409] (and considering the monochromatic case[410]), we can multiple the absolute value of MTF_{optics} by the one of $MTF_{detector}$.

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[408] Position is the pure spatial domain, frequency (RGB components) is the pure temporal domain in the simply hypothesis of steady state analysis, luminance is the energy/power domain. Position and frequency are independent in the non-relativistic treatment. See Note 131 for the assumption that there are not high (relative) speeds or accelerations between the scene and the observing system.

A.5.1 The limit of the information quantity received by the brain and the optical instruments seen by the eye

Optical instruments (telescope, microscope, etc.) are able to increase the density of information[411], but in a limited *f.o.v.*[412], because, totally, they can not increase the information quantity. So when seen by the eye[413], the brain information capability can not be overpassed. Then, for example, the spatial resolution can be improved, but the *f.o.v.* must be decreased consequently respect to the *f.o.v.* of the nuke eye.

For example: for a telescope... (x1000)… ; for a microscope.. (x800)....

*Fig. APP-30: With p*arameter linked to the total geometrical information quantity: *MTF* x *f.o.v.* (x eye f.o.v.)

A.5.2 Conservation of the information quantity received by the brain

As seen in this book, because the information capability[414] for a refractive-defective eye is less than an emmetropic eye, we can take a photo that, seen by the same eye but corrected, or by an emmetropic eye, gives to the brain the same information quantity than the one received by the brain of the guy with defective eye.

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[409] Ideal above all for the detector (optics is quasi-linear), see Par.*A.1.2*.

[410] Without lost of generality only in the simplified hypothesis that all systems are no-dispersive systems (in reality systems in time domain have not an ideal pulse response, so, obviously, there is lost of information also in time domain (enlargement in time of the “original”).

[411] So the spatial resolution; but in time domain infrared telescope, for example, can be able to enlarge the wavelength information respect to the nuke eye, but, again, substituting this new wavelength to another one.

[412] If we condier the information flow in time (channel rate), it can not increase. To see all the *f.o.v.* we must last time to scan the total *f.o.v.*.

[413] Finally, to see is to believe.

[414] Limitedly to the position information of the object: colour, for example, could be equal or better than a “normal” emmetropic eye.

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